

7

Process and Measurement System Capability Analysis

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- Supplemental Material for Chapter 7**
- S7-1 Fixed versus Random Factors in the Analysis of Variance
 - S7-2 More about Analysis of Variance Methods for Measurement Systems Capability Studies

The supplemental material is on the textbook website www.wiley.com/college/montgomery.

LEARNING OBJECTIVES

1. Investigate and analyze process capability using control charts, histograms, and probability plots
2. Understand the difference between process capability and process potential
3. Calculate and properly interpret process capability ratios
4. Understand the role of the normal distribution in interpreting most process capability ratios
5. Calculate confidence intervals on process capability ratios
6. Know how to conduct and analyze a measurement systems capability (or gauge R & R) experiment
7. Know how to estimate the components of variability in a measurement system
8. Know how to set specifications on components in a system involving interaction components to ensure that overall system requirements are met
9. Estimate the natural limits of a process from a sample of data from that process

Process Capability

Process capability refers to the **uniformity** of the process. Obviously, the variability of critical-to-quality characteristics in the process is a measure of the uniformity of output. There are two ways to think of this variability:

1. The natural or inherent variability in a critical-to-quality characteristic at a specified time; that is, “instantaneous” variability
2. The variability in a critical-to-quality characteristic over time

Natural tolerance limits are defined as follows:

$$\text{UNTL} = \mu + 3\sigma$$

$$\text{LNTL} = \mu - 3\sigma$$

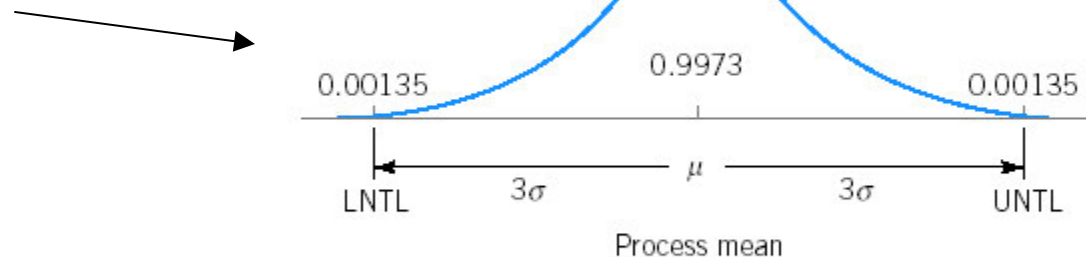


Figure 7-1 Upper and lower natural tolerance limits in the normal distribution.

We define **process capability analysis** as an engineering study to estimate process capability. The estimate of process capability may be in the form of a probability distribution having a specified shape, center (mean), and spread (standard deviation). For example, we may determine that the process output is normally distributed with mean $\mu = 1.0$ cm and standard deviation $\sigma = 0.001$ cm. In this sense, a process capability analysis may be performed **without regard to specifications on the quality characteristic**. Alternatively, we may express process capability as a percentage outside of specifications. However, specifications are not *necessary* to process capability analysis.

Uses of process capability data:

1. Predicting how well the process will hold the tolerances
2. Assisting product developers/designers in selecting or modifying a process
3. Assisting in establishing an interval between sampling for process monitoring
4. Specifying performance requirements for new equipment
5. Selecting between competing suppliers and other aspects of supply chain management
6. Planning the sequence of production processes when there is an interactive effect of processes on tolerances
7. Reducing the variability in a manufacturing process

Reasons for Poor Process Capability



Figure 7-3 Some reasons for poor process capability. (a) Poor process centering. (b) Excess process variability.

Process may have good
potential capability

7-2 PROCESS CAPABILITY ANALYSIS USING A HISTOGRAM OR A PROBABILITY PLOT

Data collection steps:

1. Choose the machine or machines to be used. If the results based on one (or a few) machines are to be extended to a larger population of machines, the machine selected should be representative of those in the population. Furthermore, if the machine has multiple workstations or heads, it may be important to collect the data so that head-to-head variability can be isolated. This may imply that designed experiments should be used.
2. Select the process operating conditions. Carefully define conditions, such as cutting speeds, feed rates, and temperatures, for future reference. It may be important to study the effects of varying these factors on process capability.
3. Select a representative operator. In some studies, it may be important to estimate *operator* variability. In these cases, the operators should be selected at random from the population of operators.
4. Carefully monitor the data-collection process, and record the time order in which each unit is produced.

Example 7-1, Glass Container Data

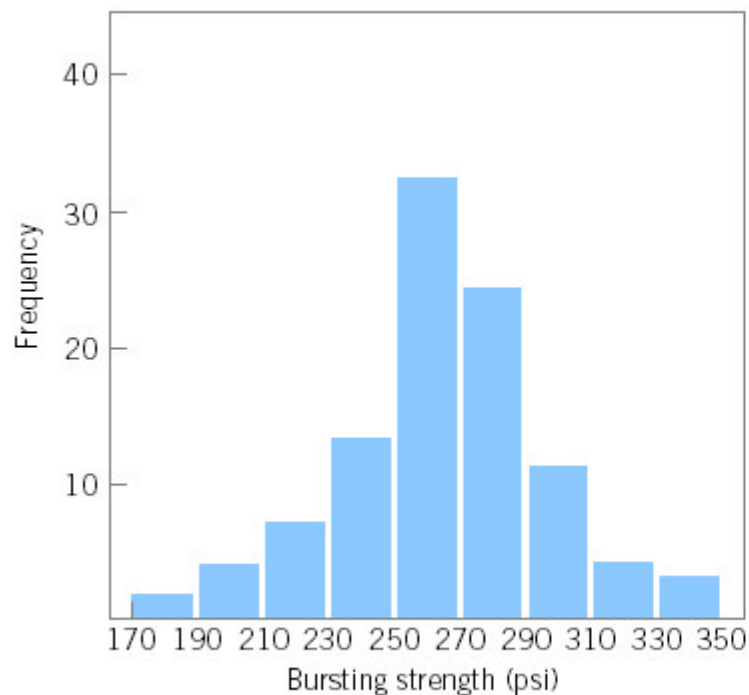


Figure 7-2 Histogram for the bursting-strength data.

Table 7-1 Bursting Strengths for 100 Glass Containers

265	197	346	280	265	200	221	265	261	278
205	286	317	242	254	235	176	262	248	250
263	274	242	260	281	246	248	271	260	265
307	243	258	321	294	328	263	245	274	270
220	231	276	228	223	296	231	301	337	298
268	267	300	250	260	276	334	280	250	257
260	281	208	299	308	264	280	274	278	210
234	265	187	258	235	269	265	253	254	280
299	214	264	267	283	235	272	287	274	269
215	318	271	293	277	290	283	258	275	251

..... **EXAMPLE 7-1**

To illustrate the use of a histogram to estimate process capability, consider Fig. 7-2, which presents a histogram of the bursting strength of 100 glass containers. The data are shown in Table 7-1. Analysis of the 100 observations gives

$$\bar{x} = 264.06 \quad s = 32.02$$

Consequently, the process capability would be estimated as

$$\bar{x} \pm 3s$$

or

$$264.06 \pm 3(32.02) \simeq 264 \pm 96 \text{ psi}$$

Furthermore, the shape of the histogram implies that the distribution of bursting strength is approximately normal. Thus, we can estimate that approximately 99.73% of the bottles manufactured by this process will burst between 168 and 360 psi. Note that we can estimate process capability *independent of the specifications on bursting strength*.

.....

Probability Plotting

To illustrate the use of a normal probability plot in a process capability study, consider the following 20 observations on glass container bursting strength: 197, 200, 215, 221, 231, 242, 245, 258, 265, 265, 271, 275, 277, 278, 280, 283, 290, 301, 318, and 346.

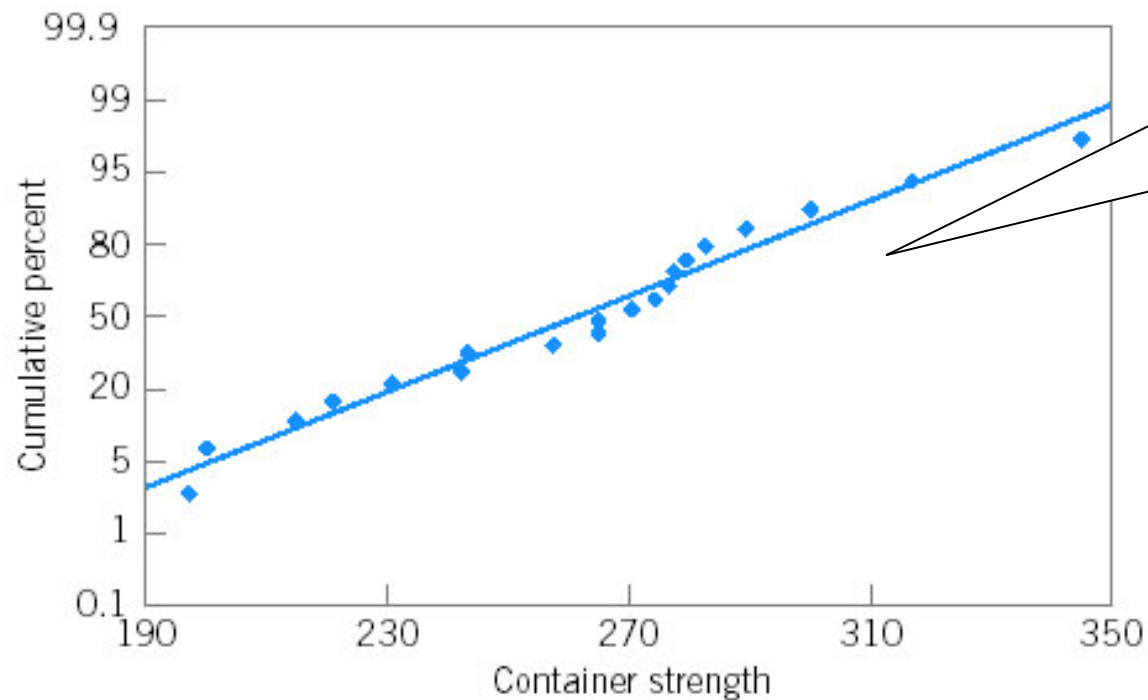


Figure 7-4 Normal probability plot of the container-strength data.

Care should be exercised in using probability plots. If the data do not come from the assumed distribution, inferences about process capability drawn from the plot may be seriously in error. Figure 7-5 presents a normal probability plot of times to failure (in hours) of a valve in a chemical plant. From examining this plot, we can see that the distribution of failure time is not normal.

An obvious disadvantage of probability plotting is that it is not an objective procedure. It is possible for two analysts to arrive at different conclusions using the same data. For this reason, it is often desirable to supplement probability plots with more formally based goodness-of-fit tests. A good introduction to these tests is in Shapiro (1980). Augmenting the interpretation of a normal probability plot with the Shapiro–Wilk test for normality can make the procedure much more powerful and objective.

- The distribution may not be normal; other types of probability plots can be useful in determining the appropriate distribution. See Chapter 2, also Figure 7-6.

7-3 PROCESS CAPABILITY RATIOS

$$C_p = \frac{USL - LSL}{6\sigma} \quad (7-4)$$

$$\hat{C}_p = \frac{USL - LSL}{6\hat{\sigma}} \quad (7-5)$$

To illustrate the calculation of C_p , recall the semiconductor hard-bake process first analyzed in Example 5-1 using $\bar{x}$ and R charts. The specifications on flow width are USL = 1.00 microns and LSL = 2.00 microns, and from the R chart we estimated $\sigma = \bar{R}/d_2 = 0.1398$. Thus, our estimate of the PCR C_p is

$$\hat{C}_p = \frac{\text{USL} - \text{LSL}}{6\hat{\sigma}} = \frac{2.00 - 1.00}{6(0.1398)} = 1.192$$

In Chapter 5, we assumed that flow width is approximately normally distributed (a reasonable assumption, based on the histogram in Fig. 7-7) and the cumulative normal distribution table in the Appendix was used to estimate that the process produces approximately 350 ppm (parts per million) defective.

Practical interpretation – relationship of PCR to the proportion of the tolerance interval used by the process:

$$P = \left(\frac{1}{C_p} \right) 100 \quad (7-6)$$

For the hard bake process:

$$P = \left(\frac{1}{1.192} \right) 100 = 83.89$$

One-Sided PCR

$$C_{pu} = \frac{USL - \mu}{3\sigma} \quad (\text{upper specification only}) \quad (7-7)$$

$$C_{pl} = \frac{\mu - LSL}{3\sigma} \quad (\text{lower specification only}) \quad (7-8)$$

Interpretation of the PCR

Table 7-2 Values of the Process Capability Ratio (C_p) and Associated Process Fallout for a Normally Distributed Process (in Defective ppm) That Is in Statistical Control

PCR	Process Fallout (in defective ppm)	
	One-Sided Specifications	Two-Sided Specifications
0.25	226,628	453,255
0.50	66,807	133,614
0.60	35,931	71,861
0.70	17,865	35,729
0.80	8,198	16,395
0.90	3,467	6,934
1.00	1,350	2,700
1.10	484	967
1.20	159	318
1.30	48	96
1.40	14	27
1.50	4	7
1.60	1	2
1.70	0.17	0.34
1.80	0.03	0.06
2.00	0.0009	0.0018

Assumptions for Interpretation of Numbers in Table 7-2

1. The quality characteristic has a normal distribution.
2. The process is in statistical control.
3. In the case of two-sided specifications, the process mean is centered between the lower and upper specification limits.

- Violation of these assumptions can lead to big trouble in using the data in Table 7-2.

Table 7-3 Recommended Minimum Values of the Process Capability Ratio

	Two-Sided Specifications	One-Sided Specifications
Existing processes	1.33	1.25
New processes	1.50	1.45
Safety, strength, or critical parameter, existing process	1.50	1.45
Safety, strength, or critical parameter, new process	1.67	1.60

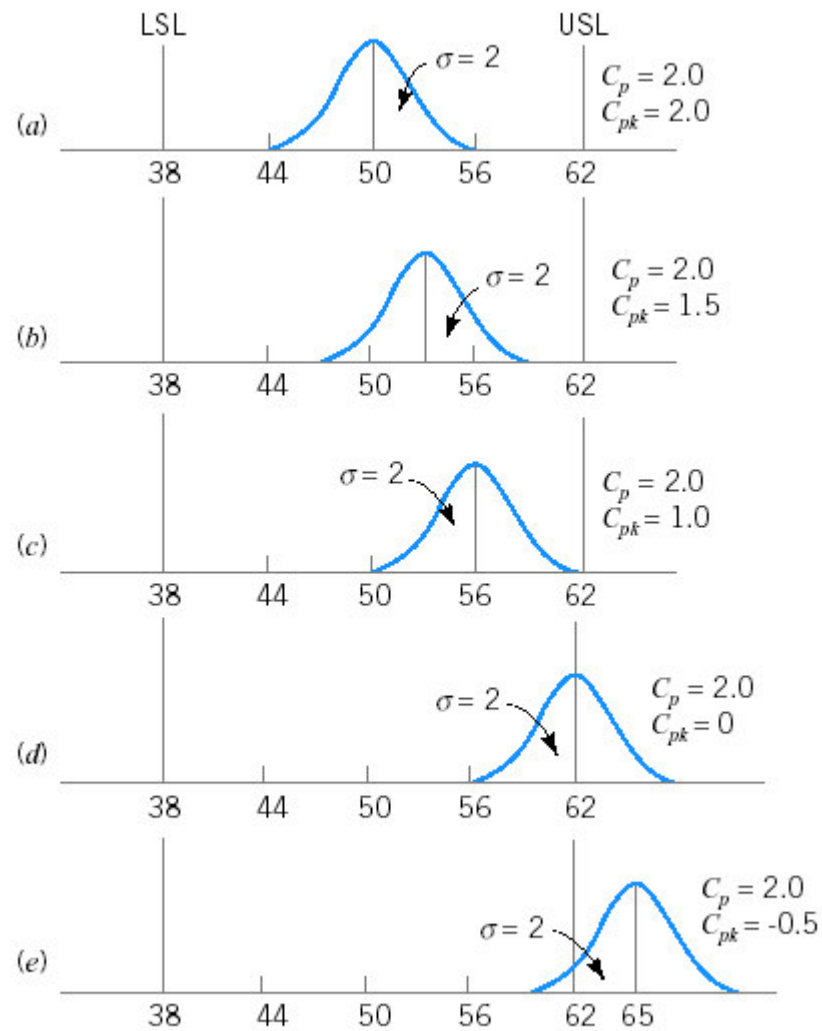


Figure 7-8 Relationship of C_p and C_{pk} .

- C_p does not take process centering into account
- It is a measure of *potential* capability, not *actual* capability

A Measure of Actual Capability

$$C_{pk} = \min(C_{pu}, C_{pl}) \quad (7-9)$$

Note that C_{pk} is simply the one-sided PCR for the specification limit nearest to the process average. For the process shown in Fig. 7-8b, we would have

$$\begin{aligned} C_{pk} &= \min(C_{pu}, C_{pl}) \\ &= \min\left(C_{pu} = \frac{USL - \mu}{3\sigma}, C_{pl} = \frac{\mu - LSL}{3\sigma}\right) \\ &= \min\left(C_{pu} = \frac{62 - 53}{3(2)} = 1.5, C_{pl} = \frac{53 - 38}{3(2)} = 2.5\right) \\ &= 1.5 \end{aligned}$$

Normality and Process Capability Ratios

- The assumption of normality is critical to the usual interpretation of these ratios (such as Table 7-2)
- For non-normal data, options are
 1. Transform non-normal data to normal
 2. Extend the usual definitions of PCR's to handle non-normal data
 3. Modify the definitions of PCR's for general families of distributions

Other Types of Process Capability Ratios (Section 7-3.5)

- First generation
- Second generation
- Third generation
- Lots of research has been done to develop ratios that overcome some of the problems with the basic ones
- Not much evidence that these ratios are used to any significant extent in practice

- Confidence intervals are an important way to express the information in a PCR

$$\hat{C}_p \sqrt{\frac{\chi^2_{1-\alpha/2, n-1}}{n-1}} \leq C_p \leq \hat{C}_p \sqrt{\frac{\chi^2_{\alpha/2, n-1}}{n-1}} \quad (7-20)$$

..... EXAMPLE 7-4

Suppose that a stable process has upper and lower specifications at $USL = 62$ and $LSL = 38$. A sample of size $n = 20$ from this process reveals that the process mean is centered approximately at the midpoint of the specification interval and that the sample standard deviation $s = 1.75$. Therefore, a point estimate of C_p is

$$\hat{C}_p = \frac{USL - LSL}{6s} = \frac{62 - 38}{6(1.75)} = 2.29$$

The 95% confidence interval on C_p is found from equation 7-20 as follows:

$$\begin{aligned}\hat{C}_p \sqrt{\frac{\chi^2_{1-0.025, n-1}}{n-1}} &\leq C_p \leq \hat{C}_p \sqrt{\frac{\chi^2_{0.025, n-1}}{n-1}} \\ 2.29 \sqrt{\frac{8.91}{19}} &\leq C_p \leq 2.29 \sqrt{\frac{32.85}{19}} \\ 1.57 &\leq C_p \leq 3.01\end{aligned}$$

CI is wide
because the
sample size is
small

where $\chi^2_{0.975, 19} = 8.91$ and $\chi^2_{0.025, 19} = 32.85$ were taken from Appendix Table III.

$$\hat{C}_{pk} \left[1 - Z_{\alpha/2} \sqrt{\frac{1}{9n\hat{C}_{pk}^2} + \frac{1}{2(n-1)}} \right] \leq C_{pk} \\ \leq \hat{C}_{pk} \left[1 + Z_{\alpha/2} \sqrt{\frac{1}{9n\hat{C}_{pk}^2} + \frac{1}{2(n-1)}} \right] \quad (7-21)$$

Kotz and Lovelace (1998) give an extensive summary of confidence intervals for various PCR's.

..... **EXAMPLE 7-5**

A sample of size $n = 20$ from a stable process is used to estimate C_{pk} , with the result that $\hat{C}_{pk} = 1.33$. Using equation 7-21, an approximate 95% confidence interval on C_{pk} is

$$\begin{aligned} \hat{C}_{pk} \left[1 - Z_{\alpha/2} \sqrt{\frac{1}{9n\hat{C}_{pk}^2} + \frac{1}{2(n-1)}} \right] \\ \leq C_{pk} \leq \hat{C}_{pk} \left[1 + Z_{\alpha/2} \sqrt{\frac{1}{9n\hat{C}_{pk}^2} + \frac{1}{2(n-1)}} \right] \\ 1.33 \left[1 - 1.96 \sqrt{\frac{1}{9(20)(1.33)^2} + \frac{1}{2(19)}} \right] \\ \leq C_{pk} \leq 1.33 \left[1 + 1.96 \sqrt{\frac{1}{9(20)(1.33)^2} + \frac{1}{2(19)}} \right] \end{aligned}$$

or

$$0.88 \leq C_{pk} \leq 1.78$$

CI is wide
because the
sample size is
small

process performance indices P_p and P_{pk} , where, for example,

$$\hat{P}_p = \frac{\text{USL} - \text{LSL}}{6s}$$

Recommended for use when the process is not in control.

Now it is clear that when the process is normally distributed and in control, $\hat{P}_p$ is essentially $\hat{C}_p$ and $\hat{P}_{pk}$ is essentially $\hat{C}_{pk}$ because for a stable process the difference between s and $\hat{\sigma} = \bar{R}/d_2$ is minimal. However, please note that if the process is **not** in control, the indices P_p and P_{pk} have no meaningful interpretation relative to process capability, because they cannot predict process performance. Furthermore, their statistical properties are not determinable, and so no valid inference can be made regarding their true (or population) values. Also, P_p and P_{pk} provide no motivation or incentive to the companies that use them to bring their processes into control.

Kotz and Lovelace (1998) strongly recommend **against** the use of P_p and P_{pk} , indicating that these indices are actually a step **backwards** in quantifying process capability. They refer to the mandated use of P_p and P_{pk} through quality standards or industry guidelines as undiluted **statistical terrorism** (i.e., the use or misuse of statistical methods along with threats and/or intimidation to achieve a business objective).

This author agrees completely with Kotz and Lovelace. The process performance indices P_p and P_{pk} are actually more than a step backwards. **They are a waste of engineering and management effort—they tell you nothing.** Unless the process is stable (in control), no index is going to carry useful predictive information about process capability or convey any information about future performance. Instead of imposing the use of meaningless indices, organizations should devote effort to developing and implementing an effective process characterization and control plan. The U.S. semiconductor industry did this in the late 1980s (at Sematech) with great success. This is a much more reasonable and effective approach to process improvement.

7-4 PROCESS CAPABILITY ANALYSIS USING A CONTROL CHART

Table 7-5 Glass Container Strength Data (psi)

Sample	Data					$\bar{x}$	R
1	265	205	263	307	220	252.0	102
2	268	260	234	299	215	255.2	84
3	197	286	274	243	231	246.2	89
4	267	281	265	214	318	269.0	104
5	346	317	242	258	276	287.8	104
6	300	208	187	264	271	246.0	113
7	280	242	260	321	228	266.2	93
8	250	299	258	267	293	273.4	49
9	265	254	281	294	223	263.4	71
10	260	308	235	283	277	272.6	73
11	200	235	246	328	296	261.0	128
12	276	264	269	235	290	266.8	55
13	221	176	248	263	231	227.8	87
14	334	280	265	272	283	286.8	69
15	265	262	271	245	301	268.8	56
16	280	274	253	287	258	270.4	34
17	261	248	260	274	337	276.0	89
18	250	278	254	274	275	266.2	28
19	278	250	265	270	298	272.2	48
20	257	210	280	269	251	253.4	70
						$\bar{\bar{x}} = 264.06$	$\bar{\bar{R}} = 77.3$

- Specifications not needed to estimate parameters and establish stability

R Chart

Center line = $\bar{\bar{R}} = 77.3$

$$UCL = D_4 \bar{\bar{R}} = (2.115)(77.3) = 163.49$$

$$LCL = D_3 \bar{\bar{R}} = (0)(77.3) = 0$$

$\bar{x}$ Chart

Center line = $\bar{\bar{x}} = 264.06$

$$UCL = \bar{\bar{x}} + A_2 \bar{\bar{R}} = 264.06 + (0.577)(77.3) = 308.66$$

$$LCL = \bar{\bar{x}} - A_2 \bar{\bar{R}} = 264.06 - (0.577)(77.3) = 219.46$$

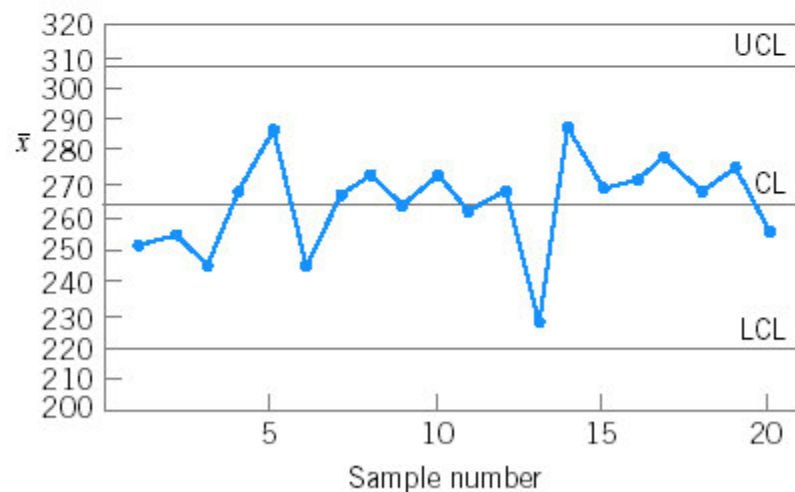
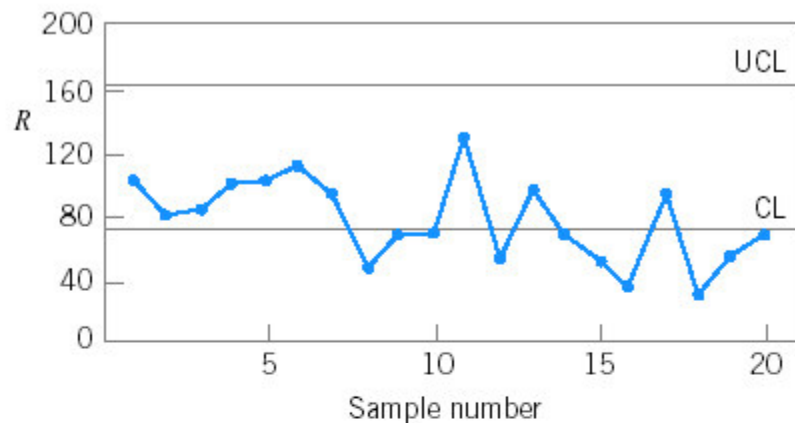


Figure 7-12 $\bar{x}$ and R charts for the bottle-strength data.

$$\hat{\mu} = \bar{\bar{x}} = 264.06$$

$$\hat{\sigma} = \frac{\bar{R}}{d_2} = \frac{77.3}{2.326} = 33.23$$

- Since $LSL = 200$

$$\hat{C}_{pl} = \frac{\mu - LSL}{3\hat{\sigma}} = \frac{264.06 - 200}{3(33.23)} = 0.64$$

7-5 PROCESS CAPABILITY ANALYSIS USING DESIGNED EXPERIMENTS

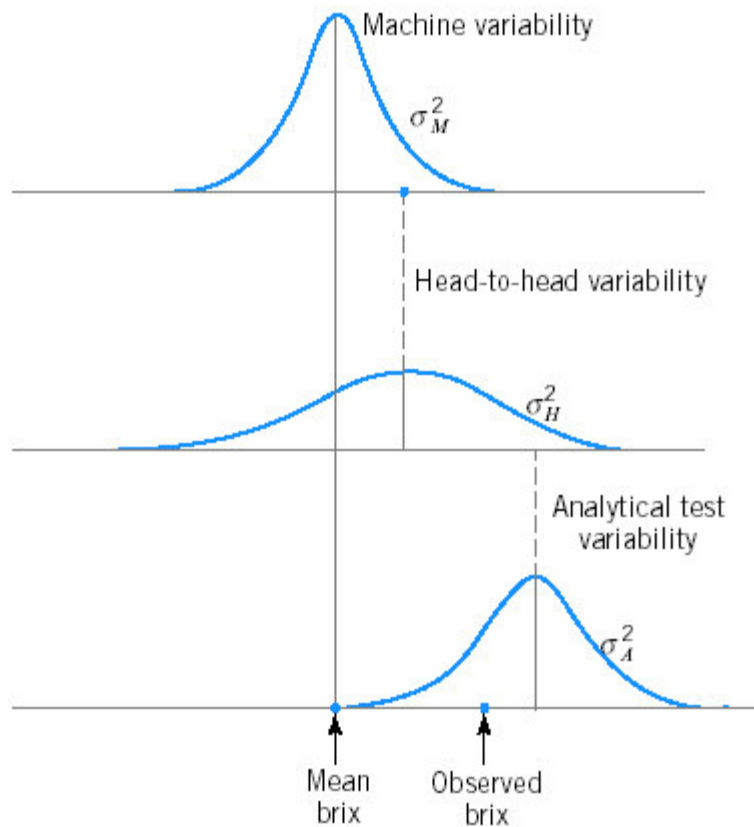


Figure 7-13 Sources of variability in the bottling line example.

- Use DOX to isolate sources of variability

7-6 GAUGE AND MEASUREMENT SYSTEM CAPABILITY STUDIES

1. Determine how much of the total observed variability is due to the gauge or instrument
2. Isolate the components of variability in the measurement system
3. Assess whether the instrument or gauge is capable (that is, is it suitable for the intended application)

A simple but reasonable model for measurement system capability studies is

$$y = x + \varepsilon \quad (7-23)$$

where y is the total observed measurement, x is the true value of the measurement on a unit of product, and ε is the measurement error. We will assume that x and ε are normally and independently distributed random variables with means μ and 0 and variances (σ_P^2) and (σ_{Gauge}^2), respectively. The variance of the total observed measurement, y , is then

$$\sigma_{\text{Total}}^2 = \sigma_P^2 + \sigma_{\text{Gauge}}^2 \quad (7-24)$$

..... **EXAMPLE 7-7**

Measuring Gauge Capability

An instrument is to be used as part of a proposed SPC implementation. The quality-improvement team involved in designing the SPC system would like to get an assessment of gauge capability. Twenty units of the product are obtained, and the process operator who will actually take the measurements for the control chart uses the instrument to measure each unit of product twice. The data are shown in Table 7-6.

Figure 7-14 shows the $\bar{x}$ and R charts for these data. Note that the $\bar{x}$ chart exhibits many out-of-control points. This is to be expected, because in this situation the $\bar{x}$ chart has an interpretation that is somewhat different from the usual interpretation. The $\bar{x}$ chart in this example shows the **discriminating power** of the instrument—literally, the ability of the gauge to distinguish between units of product. The R chart directly shows the magnitude of measurement error, or the gauge capability. The R values represent the difference between measurements made on the same unit using the same instrument. In this example, the R chart is in control. This indicates that the operator is having no difficulty in making consistent measurements. Out-of-control points on the R chart could indicate that the operator is having difficulty using the instrument.

Table 7-6 Parts Measurement Data

Part Number	Measurements		$\bar{x}$	R
	1	2		
1	21	20	20.5	1
2	24	23	23.5	1
3	20	21	20.5	1
4	27	27	27.0	0
5	19	18	18.5	1
6	23	21	22.0	2
7	22	21	21.5	1
8	19	17	18.0	2
9	24	23	23.5	1
10	25	23	24.0	2
11	21	20	20.5	1
12	18	19	18.5	1
13	23	25	24.0	2
14	24	24	24.0	0
15	29	30	29.5	1
16	26	26	26.0	0
17	20	20	20.0	0
18	19	21	20.0	2
19	25	26	25.5	1
20	19	19	19.0	0
$\bar{\bar{x}} = 22.3$			$\bar{R} = 1.0$	

$$\hat{\sigma}_{\text{Gauge}} = \frac{\bar{R}}{d_2} = \frac{1.0}{1.128} = 0.887$$

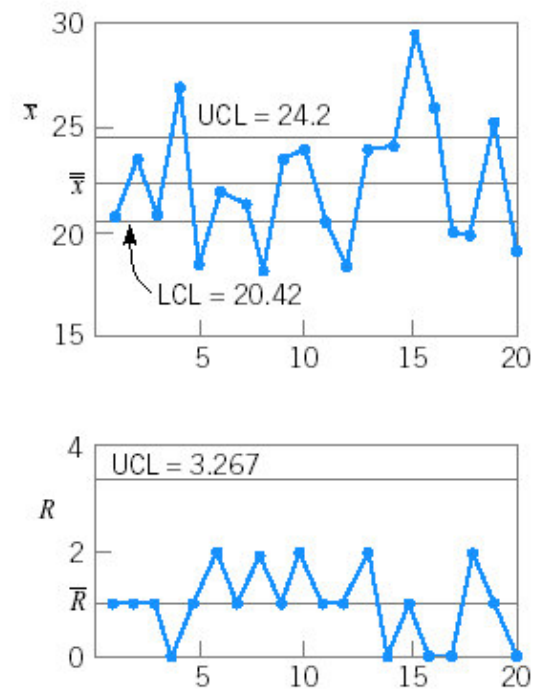


Figure 7-14 Control charts for the gauge capability analysis in Example 7-7.

$$P/T = \frac{k\hat{\sigma}_{\text{Gauge}}}{\text{USL} - \text{LSL}} \quad (7-25)$$

In equation 7-25, popular choices for the constant k are $k = 5.15$ and $k = 6$. The value $k = 5.15$ corresponds to the limiting value of the number of standard deviations between bounds of a 95% tolerance interval that contains at least 99% of a normal population, and $k = 6$ corresponds to the number of standard deviations between the usual natural tolerance limits of a normal population.

The part used in Example 7-7 has $\text{USL} = 60$ and $\text{LSL} = 5$. Therefore, taking $k = 6$ in equation 7-25, an estimate of the P/T ratio is

$$P/T = \frac{6(0.887)}{60 - 5} = \frac{5.32}{55} = 0.097$$

Values of the estimated ratio P/T of 0.1 or less often are taken to imply adequate gauge capability. This is based on the generally used rule that requires a measurement device to be calibrated in units one-tenth as large as the accuracy required in the final measurement. However, we should use **caution** in accepting this general rule of thumb in all cases. A gauge must be sufficiently capable to measure product accurately enough and precisely enough so that the analyst can make the correct decision. This may not necessarily require that $P/T \leq 0.1$.

Estimating the Variance Components

$$\hat{\sigma}_{\text{Total}}^2 = s^2 = (3.17)^2 = 10.05$$

Since from equation 7-24 we have

$$\sigma_{\text{Total}}^2 = \sigma_P^2 + \sigma_{\text{Gauge}}^2$$

and because we have an estimate of $\hat{\sigma}_{\text{Gauge}}^2 = (0.887)^2 = 0.79$, we can obtain an estimate of σ_P^2 as

$$\hat{\sigma}_P^2 = \hat{\sigma}_{\text{Total}}^2 - \hat{\sigma}_{\text{Gauge}}^2 = 10.05 - 0.79 = 9.26$$

Therefore, an estimate of the standard deviation of the product characteristic is

$$\hat{\sigma}_P = \sqrt{9.26} = 3.04$$

There are other measures of gauge capability that have been proposed. One of these is the ratio of process (part) variability to total variability:

$$\rho_P = \frac{\sigma_P^2}{\sigma_{\text{Total}}^2} \quad (7-26)$$

and another is the ratio of measurement system variability to total variability:

$$\rho_M = \frac{\sigma_{\text{Gauge}}^2}{\sigma_{\text{Total}}^2} \quad (7-27)$$

Obviously, $\rho_P = 1 - \rho_M$. For the situation in Example 7-7 we can calculate an estimate of ρ_M as follows:

$$\hat{\rho}_M = \frac{\hat{\sigma}_{\text{Gauge}}^2}{\hat{\sigma}_{\text{Total}}^2} = \frac{0.79}{10.05} = 0.0786$$

Thus the variance of the measuring instrument contributes about 7.86% of the total observed variance of the measurements.

Another measure of measurement system adequacy is defined by the AIAG (1995) [note that there is also an updated edition of this manual, AIAG (2002)] as the **signal-to-noise ratio** (*SNR*):

$$SNR = \sqrt{\frac{2\rho_P}{1 - \rho_P}} \quad (7-28)$$

AIAG defined the SNR as the number of distinct levels or categories that can be reliably obtained from the measurements. A value of five or greater is recommended, and a value of less than two indicates inadequate gauge capability. For Example 7-7 we have $\hat{\rho}_M = 0.0786$, and using $\hat{\rho}_P = 1 - \hat{\rho}_M$ we find that $\hat{\rho}_P = 1 - \hat{\rho}_M = 0.9214 = 1 - 0.0786$, so an estimate of the *SNR* in equation 7-28 is

$$\widehat{SNR} = \sqrt{\frac{2\hat{\rho}_P}{1 - \hat{\rho}_P}} = \sqrt{\frac{2(0.9214)}{1 - 0.9214}} = 4.84$$

The gauge is not capable by this criterion

Discrimination Ratio

$$DR = \frac{1 + \rho_P}{1 - \rho_P} \quad (7-29)$$

Some authors have suggested that for a gauge to be capable the DR must exceed four. This is a very arbitrary requirement. For the situation in Example 7-7, we would calculate an estimate of the discrimination ratio as

$$\widehat{DR} = \frac{1 + \rho_P}{1 - \rho_P} = \frac{1 + 0.9214}{1 - 0.9214} = 24.45$$

Clearly by this measure, the gauge is capable.

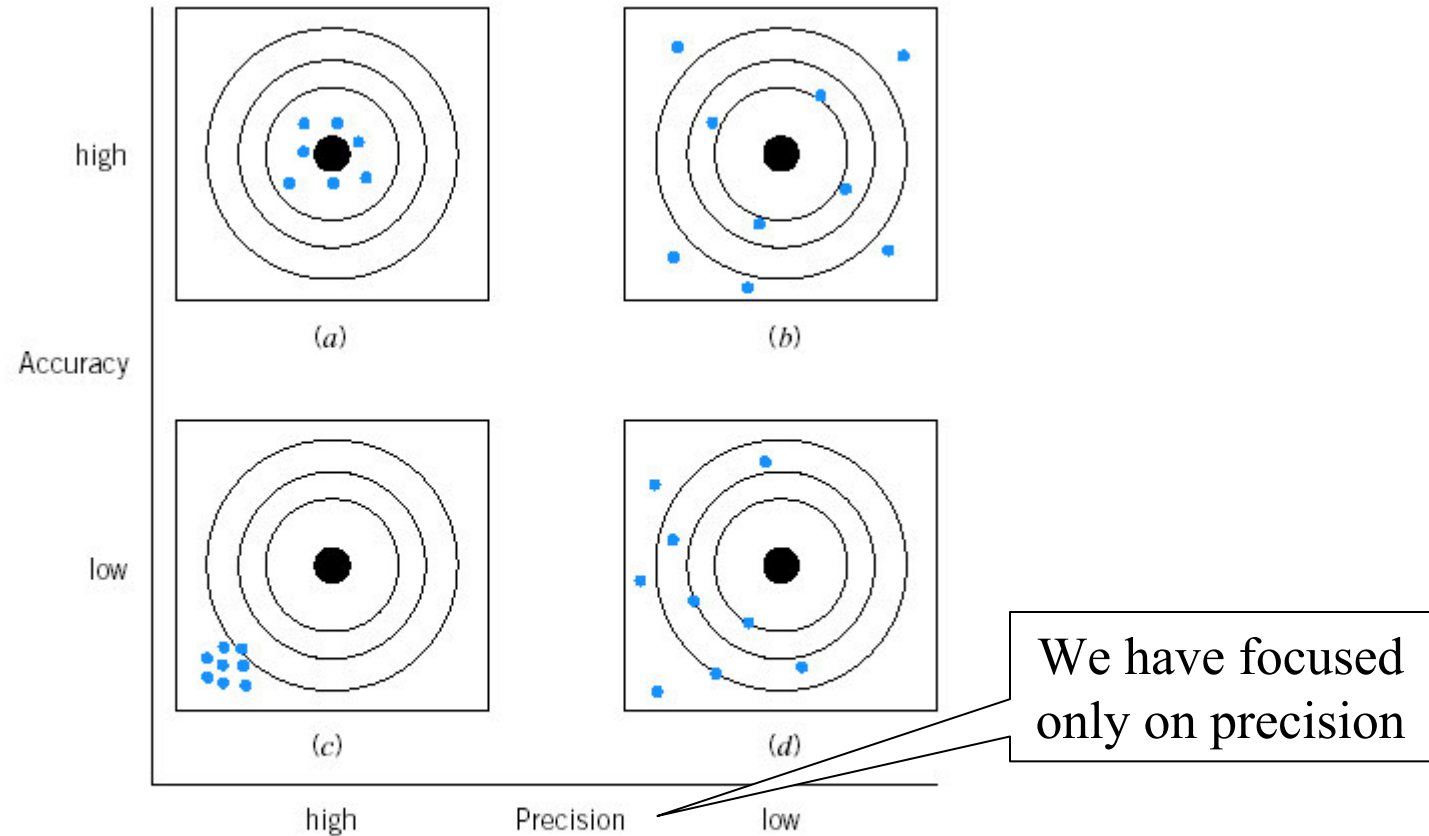


Figure 7-15 The concepts of accuracy and precision. (a) The gauge is accurate and precise. (b) The gauge is accurate but not precise. (c) The gauge is not accurate but it is precise. (d) The gauge is neither accurate nor precise.

Gauge R&R Studies

$$\sigma_{\text{Measurement Error}}^2 = \sigma_{\text{Gauge}}^2 = \sigma_{\text{Repeatability}}^2 + \sigma_{\text{Reproducibility}}^2 \quad (7-30)$$

- Gauge R&R studies are usually conducted with a factorial experiment

If there are a randomly selected parts and b randomly selected operators, and each operator measures every part n times, then the measurements (i = part, j = operator, k = measurement) could be represented by the model

$$y_{ijk} = \mu + P_i + O_j + (PO)_{ij} + \varepsilon_{ijk} \quad \begin{cases} i = 1, 2, \dots, p \\ j = 1, 2, \dots, o \\ k = 1, 2, \dots, n \end{cases} \quad (7-30)$$

where the model parameters P_i , O_j , $(PO)_{ij}$, and ε_{ijk} are all independent random variables that represent the effects of parts, operators, the interaction or joint effects of parts and operators, and random error. This is a **random effects model** analysis of variance. It is also sometimes called the standard model for a gauge R & R experiment. We assume that the random variables P_i , O_j , $(PO)_{ij}$, and ε_{ijk} are normally distributed with mean zero and variances given by $V(P_i) = \sigma_P^2$, $V(O_j) = \sigma_O^2$, $V[(PO)_{ij}] = \sigma_{PO}^2$, and $V(\varepsilon_{ijk}) = \sigma^2$. Therefore, the variance of any observation is

$$V(y_{ijk}) = \sigma_P^2 + \sigma_O^2 + \sigma_{PO}^2 + \sigma^2$$

Variance
Components

Table 7-7 Thermal Impedance Data ($^{\circ}\text{C}/\text{W} \times 100$) for the Gauge R & R Experiment

Part Number	Inspector 1			Inspector 2			Inspector 3		
	Test 1	Test 2	Test 3	Test 1	Test 2	Test 3	Test 1	Test 2	Test 3
1	37	38	37	41	41	40	41	42	41
2	42	41	43	42	42	42	43	42	43
3	30	31	31	31	31	31	29	30	28
4	42	43	42	43	43	43	42	42	42
5	28	30	29	29	30	29	31	29	29
6	42	42	43	45	45	45	44	46	45
7	25	26	27	28	28	30	29	27	27
8	40	40	40	43	42	42	43	43	41
9	25	25	25	27	29	28	26	26	26
10	35	34	34	35	35	34	35	34	35

- This is a two-factor factorial experiment
- ANOVA methods are used to conduct the R&R analysis

$$SS_{\text{Total}} = SS_{\text{Parts}} + SS_{\text{Operators}} + SS_{P \times O} + SS_{\text{Error}} \quad (7-32)$$

$$MS_P = \frac{SS_{\text{Parts}}}{p - 1}$$

$$E(MS_P) = \sigma^2 + n\sigma_{PO}^2 + bn\sigma_P^2$$

$$MS_O = \frac{SS_{\text{Operators}}}{o - 1}$$

$$E(MS_O) = \sigma^2 + n\sigma_{PO}^2 + an\sigma_O^2$$

$$MS_{PO} = \frac{SS_{P \times O}}{(p - 1)(o - 1)}$$

$$E(MS_{PO}) = \sigma^2 + n\sigma_{PO}^2$$

$$MS_E = \frac{SS_{\text{Error}}}{po(n - 1)}$$

$$E(MS_E) = \sigma^2$$

$$\hat{\sigma}^2 = MS_E$$

$$\hat{\sigma}_{PO}^2 = \frac{MS_{PO} - MS_E}{n}$$

$$\hat{\sigma}_O^2 = \frac{MS_O - MS_{PO}}{pn}$$

$$\hat{\sigma}_P^2 = \frac{MS_P - MS_{PO}}{on}$$

Table 7-8 ANOVA: Thermal Impedance versus Part Number, Operator

Factor	Type	Levels	Values						
Part Num	random	10	1	2	3	4	5	6	7
			8	9	10				
Operator	random	3	1	2	3				
Analysis of Variance for Thermal									
Source		DF	SS	MS	F		P		
Part Num		9	3935.96	437.33	162.27		0.000		
Operator		2	39.27	19.63	7.28		0.005		
Part Num*Operator		18	48.51	2.70	5.27		0.000		
Error		60	30.67	0.51					
Total		89	4054.40						
Source									
		Variance component	Error term	Expected Mean Square for Each Term (using unrestricted model)					
1 Part Num		48.2926	3	(4) + 3(3) + 9(1)					
2 Operator		0.5646	3	(4) + 3(3) + 30(2)					
3 Part Num*Operator		0.7280	4	(4) + 3(3)					
4 Error		0.5111		(4)					

$$\hat{\sigma}_P^2 = \frac{437.33 - 2.70}{(3)(3)} = 48.29$$

$$\hat{\sigma}_O^2 = \frac{19.63 - 2.70}{(10)(3)} = 0.56$$

$$\hat{\sigma}_{PO}^2 = \frac{2.70 - 0.51}{3} = 0.73$$

$$\sigma^2 = 0.51$$

- Negative estimates of a variance component would lead to fitting a reduced model, such as, for example:

$$y_{ijk} = \mu + P_i + O_j + \varepsilon_{ijk}$$

Typically we think of σ^2 as the **repeatability** variance component, and the gauge **reproducibility** as the sum of the operator and the part $\times$ operator variance components,

$$\sigma_{\text{Reproducibility}}^2 = \sigma_O^2 + \sigma_{PO}^2$$

Therefore

$$\sigma_{\text{Gauge}}^2 = \sigma_{\text{Reproducibility}}^2 + \sigma_{\text{Repeatability}}^2$$

For this Example

$$\begin{aligned}\hat{\sigma}_{\text{Gauge}}^2 &= \hat{\sigma}^2 + \hat{\sigma}_O^2 + \hat{\sigma}_{PO}^2 \\ &= 0.51 + 0.56 + 0.73 \\ &= 1.80\end{aligned}$$

The lower and upper specifications on this power module are $LSL = 18$ and $USL = 58$. Therefore the P/T ratio for the gauge is estimated as

$$\widehat{P/T} = \frac{6\hat{\sigma}_{\text{Gauge}}}{USL - LSL} = \frac{6(1.34)}{58 - 18} = 0.27$$

By the standard measures of gauge capability, this gauge would not be considered capable because the estimate of the P/T ratio exceeds 0.10.

Other Topics in Gauge R&R Studies

- Section 7-6.3 provides a description of methods to obtain confidence intervals on the variance components and measures of gauge R&R
- Section 7-6.4 presents a new measure of gauge capability, the probabilities of misclassification of parts
 - Rejecting good units (producer's risk)
 - Passing bad units (consumer's risk)
 - Methods for calculating these two probabilities are given

7-7 SETTING SPECIFICATION LIMITS ON DISCRETE COMPONENTS

Linear Combinations

In many cases, the dimension of an item is a linear combination of the dimensions of the component parts. That is, if the dimensions of the components are $x_1, x_2, \dots, x_n$, then the dimension of the final assembly is

$$y = a_1x_1 + a_2x_2 + \dots + a_nx_n \quad (7-38)$$

If the x_i are normally and independently distributed with mean μ_i and variance σ_i^2 , then y is normally distributed with mean $\mu_y = \sum_{i=1}^n a_i\mu_i$ and variance $\sigma_y^2 = \sum_{i=1}^n a_i^2 \sigma_i^2$. Therefore, if μ_i and σ_i^2 are known for each component, the fraction of assembled items falling outside the specifications can be determined.

..... **EXAMPLE 7-8**

A linkage consists of four components as shown in Fig. 7-17. The lengths of x_1, x_2, x_3 , and x_4 are known to be $x_1 \sim N(2.0, 0.0004)$, $x_2 \sim N(4.5, 0.0009)$, $x_3 \sim N(3.0, 0.0004)$, and $x_4 \sim N(2.5, 0.0001)$. The lengths of the components can be assumed independent, because they are produced on different machines. All lengths are in inches.

The design specifications on the length of the assembled linkage are 12.00 ± 0.10 . To find the fraction of linkages that fall within these specification limits, note that y is normally distributed with mean

$$\mu_y = 2.0 + 4.5 + 3.0 + 2.5 = 12.0$$

and variance

$$\sigma_y^2 = 0.0004 + 0.0009 + 0.0004 + 0.0001 = 0.0018$$

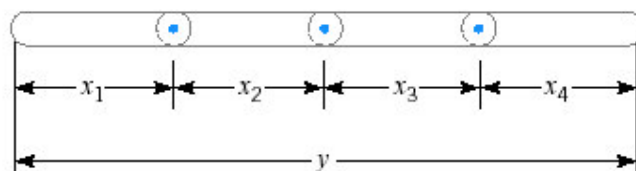


Figure 7-17 A linkage assembly with four components.

To find the fraction of linkages that are within specification, we must evaluate

$$\begin{aligned}P\{11.90 \leq y \leq 12.10\} &= P\{y \leq 12.10\} - P\{y \leq 11.90\} \\&= \Phi\left(\frac{12.10 - 12.00}{\sqrt{0.0018}}\right) - \Phi\left(\frac{11.90 - 12.00}{\sqrt{0.0018}}\right) \\&= \Phi(2.36) - \Phi(-2.36) \\&= 0.99086 - 0.00914 \\&= 0.98172\end{aligned}$$

Therefore, we conclude that 98.172% of the assembled linkages will fall within the specification limits.



In some problems, the dimension of interest may be a **nonlinear function** of the n component dimensions $x_1, x_2, \dots, x_n$ —say,

$$y = g(x_1, x_2, \dots, x_n) \quad (7-41)$$

In problems of this type, the usual approach is to approximate the nonlinear function g by a linear function of the x_i in the region of interest. If $\mu_1, \mu_2, \dots, \mu_n$ are the nominal dimensions associated with the components $x_1, x_2, \dots, x_n$, then by expanding the right-hand side of equation 7-41 in a Taylor series about $\mu_1, \mu_2, \dots, \mu_n$, we obtain

$$\begin{aligned} y &= g(x_1, x_2, \dots, x_n) \\ &= g(\mu_1, \mu_2, \dots, \mu_n) + \sum_{i=1}^n (x_i - \mu_i) \left. \frac{\partial g}{\partial x_i} \right|_{\mu_1, \mu_2, \dots, \mu_n} + R \end{aligned} \quad (7-42)$$

where R represents the higher-order terms. Neglecting the terms of higher order, we can apply the expected value and variance operators to obtain

$$\mu_y \simeq g(\mu_1, \mu_2, \dots, \mu_n) \quad (7-43)$$

and

$$\sigma_y^2 \simeq \sum_{i=1}^n \left(\left. \frac{\partial g}{\partial x_i} \right|_{\mu_1, \mu_2, \dots, \mu_n} \right)^2 \sigma_i^2 \quad (7-44)$$

This procedure to find an approximate mean and variance of a nonlinear combination of random variables is sometimes called the **delta method**. Equation 7-44 is often called the **transmission of error formula**.

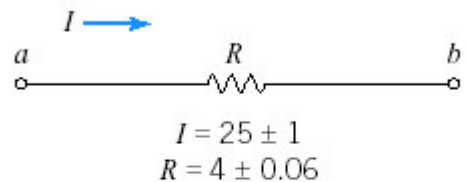
..... EXAMPLE 7-11

Consider the simple DC circuit components shown in Fig. 7-20. Suppose that the voltage across the points (a, b) is required to be 100 ± 2 V. The specifications on the current and the resistance in the circuit are shown in Fig. 7-20. We assume that the component random variables I and R are normally and independently distributed with means equal to their nominal values.

From Ohm's law, we know that the voltage is

$$V = IR$$

Since this involves a nonlinear combination, we expand V in a Taylor series about mean current μ_I and mean resistance μ_R , yielding



$$V \simeq \mu_I \mu_R + (I - \mu_I) \mu_R + (R - \mu_R) \mu_I$$

$I = 25 \pm 1$
 $R = 4 \pm 0.06$

Figure 7-20 Electrical circuit for Example 7-11.

$$\mu_V \simeq \mu_I \mu_R$$

$$\sigma_V^2 \simeq \mu_R^2 \sigma_I^2 + \mu_I^2 \sigma_R^2$$

Now suppose that I and R are centered at their nominal values and that the natural tolerance limits are defined so that $\alpha = 0.0027$ is the fraction of values of each component falling outside these limits. Assume also that the specification limits are exactly equal to the natural tolerance limits. For the current I we have $I = 25 \pm 1$ A. That is, $24 \leq I \leq 26$ A correspond to the natural tolerance limits *and* the specifications. Since $I \sim N(25, \sigma_I^2)$, and since $Z_{\alpha/2} = Z_{0.00135} = 3.00$, we have

$$\frac{26 - 25}{\sigma_I} = 3.00$$

or $\sigma_I = 0.33$. For the resistance, we have $R = 4 \pm 0.06$ ohm as the specification limits *and* the natural tolerance limits. Thus,

$$\frac{4.06 - 4.00}{\sigma_R} = 3.00$$

and $\sigma_R = 0.02$. Note that σ_I and σ_R are the largest possible values of the component standard deviations consistent with the natural tolerance limits falling inside or equal to the specification limits.

Using these results, and if we assume that the voltage V is approximately normally distributed, then

$$\mu_V \simeq \mu_I \mu_R = (25)(4) = 100 \text{ V}$$

and

$$\sigma_V^2 \simeq \mu_R^2 \sigma_I^2 + \mu_I^2 \sigma_R^2 = (4)^2 (0.33)^2 + (25)^2 (0.02)^2 = 1.99$$

approximately. Thus $\sigma_V = \sqrt{1.99} = 1.41$. Therefore, the probability that the voltage will fall within the design specifications is

$$\begin{aligned} P\{98 \leq V \leq 102\} &= P\{V \leq 102\} - P\{V \leq 98\} \\ &= \Phi\left(\frac{102 - 100}{1.41}\right) - \Phi\left(\frac{98 - 100}{1.41}\right) \\ &= \Phi(1.42) - \Phi(-1.42) \\ &= 0.92219 - 0.07781 \\ &= 0.84438 \end{aligned}$$

7-8 ESTIMATING THE NATURAL TOLERANCE LIMITS OF A PROCESS

For the normal distribution with unknown mean and variance:

$$\bar{x} \pm Z_{\alpha/2} s \quad (7-45)$$

- Difference between tolerance limits and confidence limits
- Nonparametric tolerance limits can also be calculated

..... **EXAMPLE 7-12**

The manufacturer of a solid-fuel rocket propellant is interested in finding the tolerance limits of the process such that 95% of the burning rates will lie within these limits with probability 0.99. It is known from previous experience that the burning rate is normally distributed. A random sample of 25 observations shows that the sample mean and variance of burning rate are $\bar{x} = 40.75$ and $s^2 = 1.87$, respectively. Since $\alpha = 0.05$, $\gamma = 0.99$, and $n = 25$, we find $K = 2.972$ from Appendix Table VII. Therefore, the required tolerance limits are found as $\bar{x} \pm 2.972s = 40.75 \pm (2.972)(1.37) = 40.75 \pm 4.07 = [36.68, 44.82]$.

.....

IMPORTANT TERMS AND CONCEPTS

ANOVA approach to a gauge	Natural tolerance limits of a process
R & R experiment	Nonparametric tolerance limits
Components of gauge error	Normal distribution and
Components of measurement error	process capability ratios
Confidence intervals for gauge R & R studies	One-sided process capability ratios
Confidence intervals on	P/T ratio
process capability ratios	PCR C_p
Consumer's risk or missed fault for a gauge	PCR C_{pk}
Control charts and process capability analysis	PCR C_{pm}
Delta method	Precision versus accuracy of a gauge
Discrimination ratio DR for a gauge	Process capability
Estimating variance components	Process capability analysis
Factorial experiment	Process performance indices P_p and P_{pk}
Gauge R & R experiment	Producer's risk or false failure for a gauge
Graphical methods for process capability	Product characterization
analysis	Random effects model ANOVA
Measurement systems capability analysis	Signal-to-noise ratio SNR for a gauge
Natural tolerance limits for	Tolerance stack-up problems
a normal distribution	Transmission of error formula

LEARNING OBJECTIVES

1. Investigate and analyze process capability using control charts, histograms, and probability plots
2. Understand the difference between process capability and process potential
3. Calculate and properly interpret process capability ratios
4. Understand the role of the normal distribution in interpreting most process capability ratios
5. Calculate confidence intervals on process capability ratios
6. Know how to conduct and analyze a measurement systems capability (or gauge R & R) experiment
7. Know how to estimate the components of variability in a measurement system
8. Know how to set specifications on components in a system involving interaction components to ensure that overall system requirements are met
9. Estimate the natural limits of a process from a sample of data from that process