

5

Control Charts for Variables

CHAPTER OUTLINE

- 5-1 INTRODUCTION
- 5-2 CONTROL CHARTS FOR $\bar{x}$ AND R
 - 5-2.1 Statistical Basis of the Charts
 - 5-2.2 Development and Use of $\bar{x}$ and R Charts
 - 5-2.3 Charts Based on Standard Values
 - 5-2.4 Interpretation of $\bar{x}$ and R Charts
 - 5-2.5 The Effect of Nonnormality on $\bar{x}$ and R Charts
 - 5-2.6 The Operating-Characteristic Function
 - 5-2.7 The Average Run Length for the $\bar{x}$ Chart
- 5-3 CONTROL CHARTS FOR $\bar{x}$ AND s
 - 5-3.1 Construction and Operation of $\bar{x}$ and s Charts
 - 5-3.2 The $\bar{x}$ and s Control Charts with Variable Sample Size
 - 5-3.3 The s^2 Control Chart
- 5-4 THE SHEWHART CONTROL CHART FOR INDIVIDUAL MEASUREMENTS

- 5-5 SUMMARY OF PROCEDURES FOR $\bar{x}$, R , AND s CHARTS
- 5-6 APPLICATIONS OF VARIABLES CONTROL CHARTS

Supplemental Material for Chapter 5

- S5-1 s^2 Is Not Always an Unbiased Estimator of σ^2
- S5-2 Should We Use d_2 or d_2^* in Estimating σ via the Range Method?
- S5-3 Determining When the Process Has Shifted
- S5-4 More about Monitoring Variability with Individual Observations
- S5-5 Detecting Drifts versus Shifts in the Process Mean
- S5-6 The Mean Square Successive Difference as an Estimator of σ^2

The supplemental material is on the textbook website www.wiley.com/college/montgomery.

LEARNING OBJECTIVES

1. Understand the statistical basis of Shewhart control charts for variables
2. Know how to design variables control charts
3. Know how to set up and use $\bar{x}$ and R control charts
4. Know how to estimate process capability from the control chart information
5. Know how to interpret patterns on $\bar{x}$ and R control charts
6. Know how to set up and use $\bar{x}$ and s or s^2 control charts
7. Know how to set up and use control charts for individual measurements
8. Understand the importance of the normality assumption for individuals control charts and know how to check this assumption
9. Understand the rational subgroup concept for variables control charts
10. Determine the average run length for variables control charts

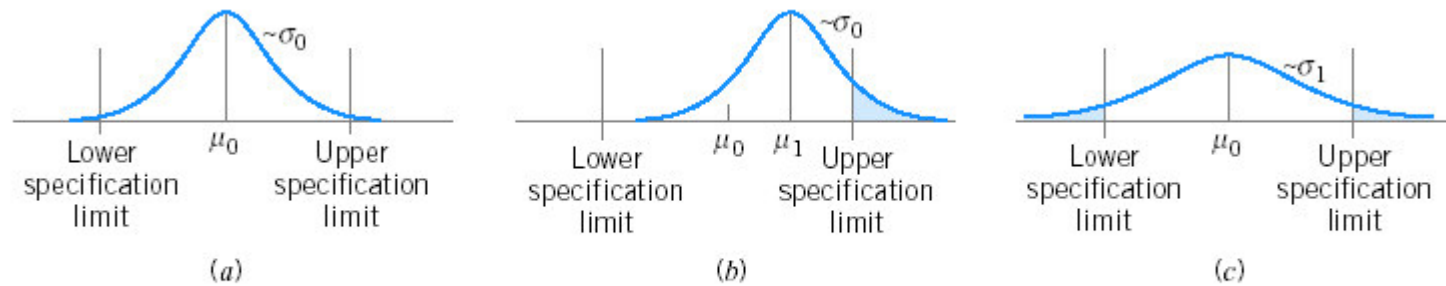


Figure 5-1 The need for controlling both process mean and process variability. (a) Mean and standard deviation at nominal levels. (b) Process mean $\mu_1 > \mu_0$. (c) Process standard deviation $\sigma_1 > \sigma_0$.

5-2 CONTROL CHARTS FOR $\bar{x}$ AND R

Suppose that a quality characteristic is normally distributed with mean μ and standard deviation σ , where both μ and σ are known. If $x_1, x_2, \dots, x_n$ is a sample of size n , then the average of this sample is

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}$$

and we know that $\bar{x}$ is normally distributed with mean μ and standard deviation $\sigma_{\bar{x}} = \sigma/\sqrt{n}$. Furthermore, the probability is $1 - \alpha$ that any sample mean will fall between

$$\mu + Z_{\alpha/2}\sigma_{\bar{x}} = \mu + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad \text{and} \quad \mu - Z_{\alpha/2}\sigma_{\bar{x}} = \mu - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad (5-1)$$

Subgroup Data with Unknown μ and σ

$$\bar{\bar{x}} = \frac{\bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_m}{m} \quad (5-2)$$

$$R = x_{\max} - x_{\min}$$

Let $R_1, R_2, \dots, R_m$ be the ranges of the m samples. The average range is

$$\bar{R} = \frac{R_1 + R_2 + \cdots + R_m}{m} \quad (5-3)$$

Control Limits for the $\bar{x}$ Chart

$$\begin{aligned} \text{UCL} &= \bar{\bar{x}} + A_2 \bar{R} \\ \text{Center line} &= \bar{\bar{x}} \\ \text{LCL} &= \bar{\bar{x}} - A_2 \bar{R} \end{aligned} \tag{5-4}$$

The constant A_2 is tabulated for various sample sizes in Appendix Table VI.

Control Limits for the R Chart

$$\begin{aligned} \text{UCL} &= D_4 \bar{R} \\ \text{Center line} &= \bar{R} \\ \text{LCL} &= D_3 \bar{R} \end{aligned} \tag{5-5}$$

The constants D_3 and D_4 are tabulated for various values of n in Appendix Table VI.

$$\hat{\sigma} = \frac{\bar{R}}{d_2} \quad (5-6)$$

If we use $\bar{\bar{x}}$ as an estimator of μ and $\bar{R}/d_2$ as an estimator of σ , then the parameters of the $\bar{x}$ chart are

$$\begin{aligned} \text{UCL} &= \bar{\bar{x}} + \frac{3}{d_2\sqrt{n}} \bar{R} \\ \text{Center line} &= \bar{\bar{x}} \\ \text{LCL} &= \bar{\bar{x}} - \frac{3}{d_2\sqrt{n}} \bar{R} \end{aligned} \quad (5-7)$$

If we define

$$A_2 = \frac{3}{d_2\sqrt{n}} \quad (5-8)$$

then equation 5-7 reduces to equation 5-4.

Now consider the R chart. The center line will be $\bar{R}$. To determine the control limits, we need an estimate of σ_R . Assuming that the quality characteristic is normally distributed, $\hat{\sigma}_R$ can be found from the distribution of the relative range $W = R/\sigma$. The standard deviation of W , say d_3 , is a known function of n . Thus, since

$$R = W\sigma$$

the standard deviation of R is

$$\sigma_R = d_3\sigma$$

Since σ is unknown, we may estimate σ_R by

$$\hat{\sigma}_R = d_3 \frac{\bar{R}}{d_2} \quad (5-9)$$

Consequently, the parameters of the R chart with the usual three-sigma control limits are

$$\begin{aligned} \text{UCL} &= \bar{R} + 3\hat{\sigma}_R = \bar{R} + 3d_3 \frac{\bar{R}}{d_2} \\ \text{Center line} &= \bar{R} \\ \text{LCL} &= \bar{R} - 3\hat{\sigma}_R = \bar{R} - 3d_3 \frac{\bar{R}}{d_2} \end{aligned} \tag{5-10}$$

If we let

$$D_3 = 1 - 3 \frac{d_3}{d_2} \quad \text{and} \quad D_4 = 1 + 3 \frac{d_3}{d_2}$$

equation 5-10 reduces to equation 5-5.

Phase I Application of $\bar{x}$ and R Charts

- Eqns 5-4 and 5-5 are **trial control limits**
 - Determined from m initial samples
 - Typically 20-25 subgroups of size n between 3 and 5
 - Any out-of-control points should be examined for assignable causes
 - If assignable causes are found, discard points from calculations and revise the trial control limits
 - Continue examination until all points plot in control
 - Adopt resulting trial control limits for use
 - If no assignable cause is found, there are two options
 1. Eliminate point as if an assignable cause were found and revise limits
 2. Retain point and consider limits appropriate for control
 - If there are many out-of-control points they should be examined for **patterns** that may identify underlying process problems

EXAMPLE 5-1

Table 5-1 Flow Width Measurements (microns) for the Hard-Bake Process

Sample Number	Wafers					$\bar{x}_i$	R_i
	1	2	3	4	5		
1	1.3235	1.4128	1.6744	1.4573	1.6914	1.5119	0.3679
2	1.4314	1.3592	1.6075	1.4666	1.6109	1.4951	0.2517
3	1.4284	1.4871	1.4932	1.4324	1.5674	1.4817	0.1390
4	1.5028	1.6352	1.3841	1.2831	1.5507	1.4712	0.3521
5	1.5604	1.2735	1.5265	1.4363	1.6441	1.4882	0.3706
6	1.5955	1.5451	1.3574	1.3281	1.4198	1.4492	0.2674
7	1.6274	1.5064	1.8366	1.4177	1.5144	1.5805	0.4189
8	1.4190	1.4303	1.6637	1.6067	1.5519	1.5343	0.2447
9	1.3884	1.7277	1.5355	1.5176	1.3688	1.5076	0.3589
10	1.4039	1.6697	1.5089	1.4627	1.5220	1.5134	0.2658
11	1.4158	1.7667	1.4278	1.5928	1.4181	1.5242	0.3509
12	1.5821	1.3355	1.5777	1.3908	1.7559	1.5284	0.4204
13	1.2856	1.4106	1.4447	1.6398	1.1928	1.3947	0.4470
14	1.4951	1.4036	1.5893	1.6458	1.4969	1.5261	0.2422
15	1.3589	1.2863	1.5996	1.2497	1.5471	1.4083	0.3499
16	1.5747	1.5301	1.5171	1.1839	1.8662	1.5344	0.6823
17	1.3680	1.7269	1.3957	1.5014	1.4449	1.4874	0.3589
18	1.4163	1.3864	1.3057	1.6210	1.5573	1.4573	0.3153
19	1.5796	1.4185	1.6541	1.5116	1.7247	1.5777	0.3062
20	1.7106	1.4412	1.2361	1.3820	1.7601	1.5060	0.5240
21	1.4371	1.5051	1.3485	1.5670	1.4880	1.4691	0.2185
22	1.4738	1.5936	1.6583	1.4973	1.4720	1.5390	0.1863
23	1.5917	1.4333	1.5551	1.5295	1.6866	1.5592	0.2533
24	1.6399	1.5243	1.5705	1.5563	1.5530	1.5688	0.1156
25	1.5797	1.3663	1.6240	1.3732	1.6887	1.5264	0.3224
						$\Sigma \bar{x}_i = 37.6400$	$\Sigma R_i = 8.1302$
						$\bar{\bar{x}} = 1.5056$	$\bar{R} = 0.32521$

$$\bar{R} = \frac{\sum_{i=1}^{25} R_i}{25} = \frac{8.1302}{25} = 0.32521$$

$$\text{LCL} = \bar{R}D_3 = 0.32521(0) = 0$$

$$\text{UCL} = \bar{R}D_4 = 0.32521(2.114) = 0.68749$$

$$\bar{\bar{x}} = \frac{\sum_{i=1}^{25} \bar{x}_i}{25} = \frac{37.6400}{25} = 1.5056$$

$$\text{UCL} = \bar{\bar{x}} + A_2 \bar{R} = 1.5056 + (0.577)(0.32521) = 1.69325$$

and

$$\text{LCL} = \bar{\bar{x}} - A_2 \bar{R} = 1.5056 - (0.577)(0.32521) = 1.31795$$

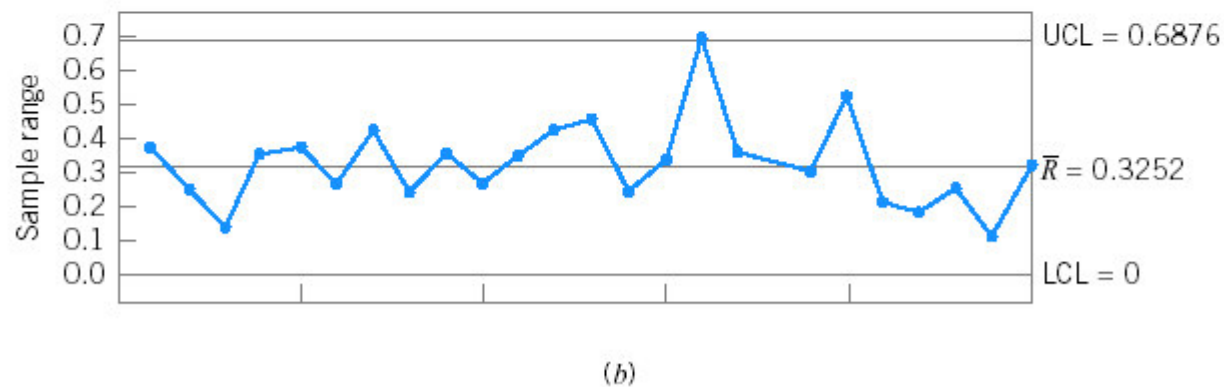
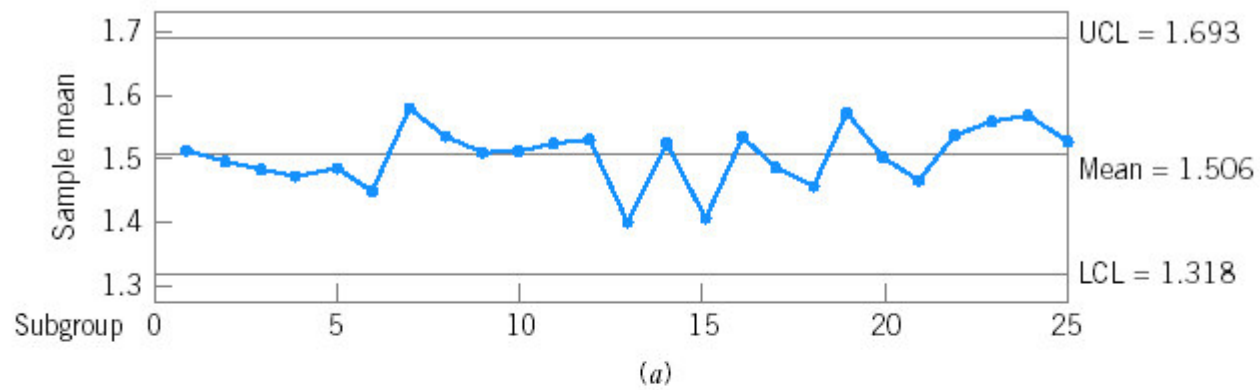


Figure 5-2 $\bar{x}$ and R charts (from Minitab) for flow width in the hard-bake process.

Estimating Process Capability

The $\bar{x}$ and R charts provide information about the performance or **capability** of the process. From the $\bar{x}$ chart, we may estimate the mean flow width of the resist in the hard-bake process as $\bar{\bar{x}} = 1.5056$ microns. The process standard deviation may be estimated using equation 5-6; that is,

$$\hat{\sigma} = \frac{\bar{R}}{d_2} = \frac{0.32521}{2.326} = 0.1398 \text{ microns}$$

where the value of d_2 for samples of size five is found in Appendix Table VI. The specification limits on flow width are 1.50 ± 0.50 microns. The control chart data may be used to describe the capability of the process to produce wafers relative to these specifications. Assuming that flow width is a normally distributed random variable, with mean 1.5056 and standard deviation 0.1398, we may estimate the fraction of nonconforming wafers produced as

$$\begin{aligned} p &= P\{x < 1.00\} + P\{x > 2.00\} \\ &= \Phi\left(\frac{1.00 - 1.5056}{0.1398}\right) + 1 - \Phi\left(\frac{2.00 - 1.5056}{0.1398}\right) \\ &= \Phi(-3.61660) + 1 - \Phi(3.53648) \\ &\simeq 0.00015 + 1 - 0.99980 \\ &\simeq 0.00035 \end{aligned}$$

That is, about 0.035 percent [350 parts per million (ppm)] of the wafers produced will be outside of the specifications.

Another way to express process capability is in terms of the **process capability ratio (PCR)** C_p , which for a quality characteristic with both upper and lower specification limits (USL and LSL, respectively) is

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma} \quad (5-11)$$

Note that the 6σ spread of the process is the basic definition of process capability. Since σ is usually unknown, we must replace it with an estimate. We frequently use $\hat{\sigma} = \bar{R}/d_2$ as an estimate of σ , resulting in an estimate $\hat{C}_p$ of C_p . For the hard-bake process, since $\bar{R}/d_2 = \hat{\sigma} = 0.1398$, we find that

$$\hat{C}_p = \frac{2.00 - 1.00}{6(0.1398)} = \frac{1.00}{0.8388} = 1.192$$

$$P = \left(\frac{1}{\hat{C}_p} \right) 100\%$$

is simply the percentage of the specification band that the process uses up. For the hard-bake process an estimate of P is

$$\hat{P} = \left(\frac{1}{\hat{C}_p} \right) 100\% = \left(\frac{1}{1.192} \right) 100\% = 83.89$$

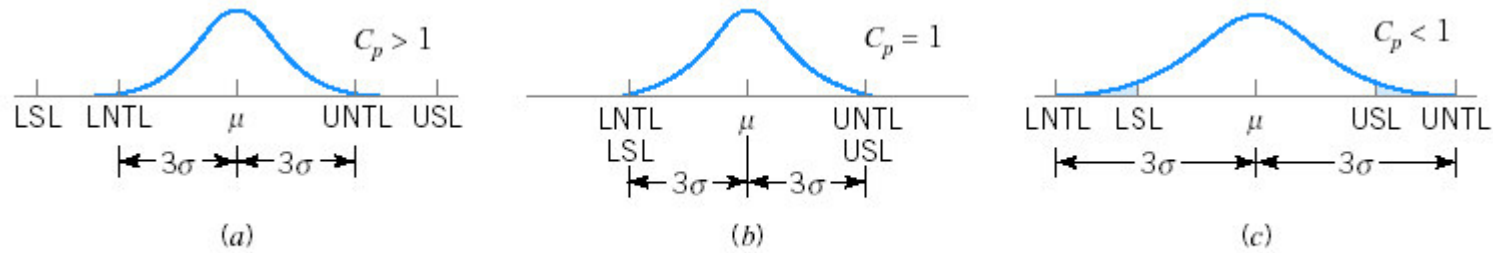


Figure 5-3 Process fallout and the process capability ratio C_p .

Revision of Control Limits and Center Lines

- Effective use of control charts requires periodic review and revision of control limits and center lines
- Sometimes users replace the center line on the $\bar{x}$ chart with a target value
- When R chart is out of control, out-of-control points are often eliminated to recompute a revised value of $\bar{R}$ which is used to determine new limits and center line on R chart and new limits on $\bar{x}$ chart

Phase II Operation of Charts

- Use of control chart for monitoring future production, once a set of reliable limits are established, is called *phase II* of control chart usage (Figure 5-4)
- A run chart showing individuals observations in each sample, called a **tolerance chart** or **tier diagram** (Figure 5-5), may reveal patterns or unusual observations in the data

Table 5-2 Additional Samples for Example 5-1

Sample Number	Wafers					$\bar{x}_i$	R_i
	1	2	3	4	5		
26	1.4483	1.5458	1.4538	1.4303	1.6206	1.4998	0.1903
27	1.5435	1.6899	1.5830	1.3358	1.4187	1.5142	0.3541
28	1.5175	1.3446	1.4723	1.6657	1.6661	1.5332	0.3215
29	1.5454	1.0931	1.4072	1.5039	1.5264	1.4152	0.4523
30	1.4418	1.5059	1.5124	1.4620	1.6263	1.5097	0.1845
31	1.4301	1.2725	1.5945	1.5397	1.5252	1.4724	0.3220
32	1.4981	1.4506	1.6174	1.5837	1.4962	1.5292	0.1668
33	1.3009	1.5060	1.6231	1.5831	1.6454	1.5317	0.3445
34	1.4132	1.4603	1.5808	1.7111	1.7313	1.5793	0.3181
35	1.3817	1.3135	1.4953	1.4894	1.4596	1.4279	0.1818
36	1.5765	1.7014	1.4026	1.2773	1.4541	1.4824	0.4241
37	1.4936	1.4373	1.5139	1.4808	1.5293	1.4910	0.0920
38	1.5729	1.6738	1.5048	1.5651	1.7473	1.6128	0.2425
39	1.8089	1.5513	1.8250	1.4389	1.6558	1.6560	0.3861
40	1.6236	1.5393	1.6738	1.8698	1.5036	1.6420	0.3662
41	1.4120	1.7931	1.7345	1.6391	1.7791	1.6716	0.3811
42	1.7372	1.5663	1.4910	1.7809	1.5504	1.6252	0.2899
43	1.5971	1.7394	1.6832	1.6677	1.7974	1.6970	0.2003
44	1.4295	1.6536	1.9134	1.7272	1.4370	1.6321	0.4839
45	1.6217	1.8220	1.7915	1.6744	1.9404	1.7700	0.3187

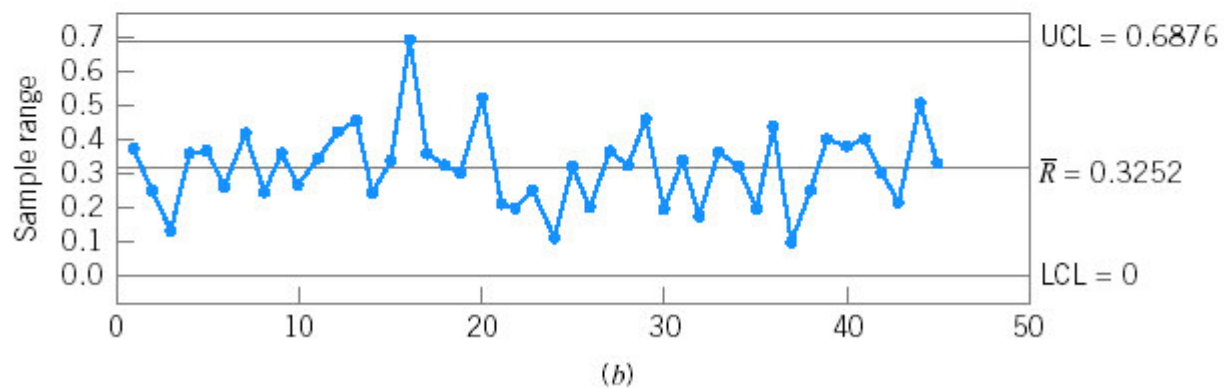
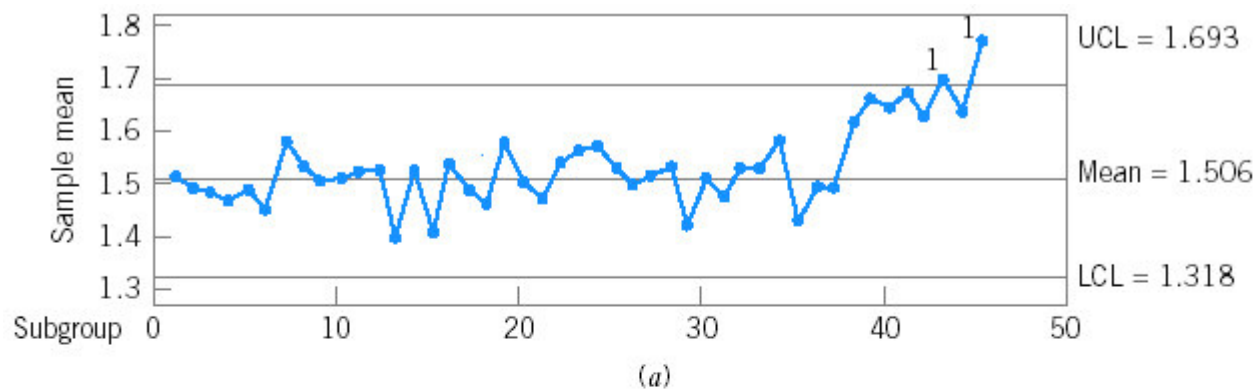


Figure 5-4 Continuation of the $\bar{x}$ and R charts in Example 5-1.

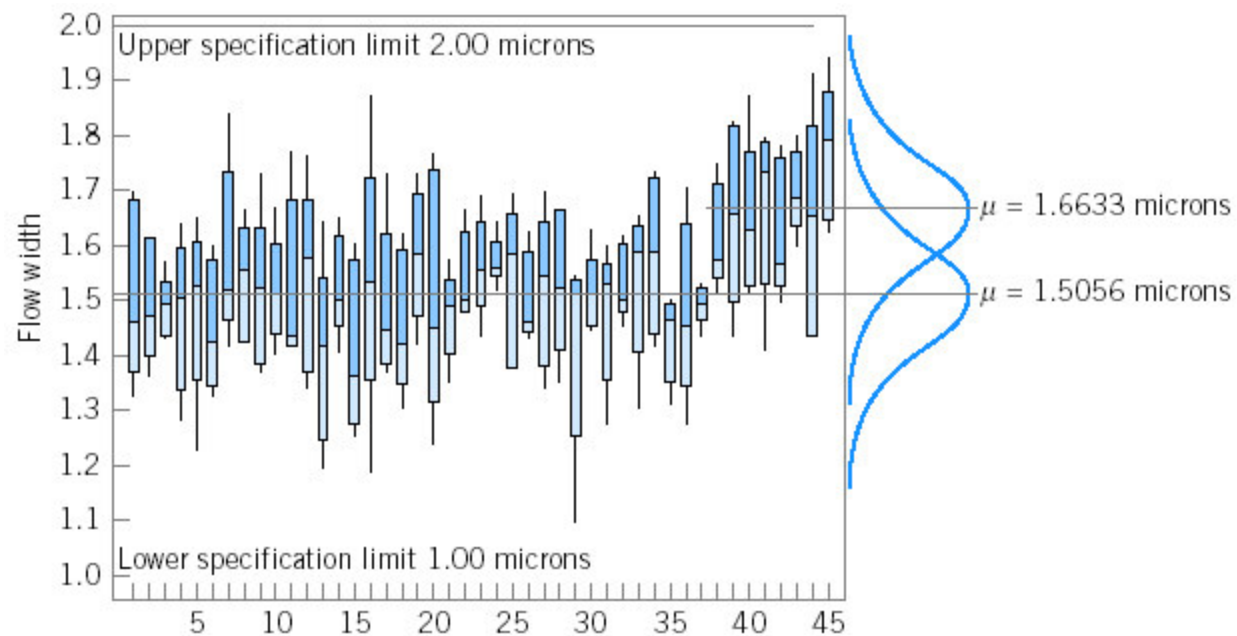


Figure 5-5 Tier chart constructed using the Minitab box plot procedure for the flow width data.

Control vs. Specification Limits

- **Control** limits are derived from natural process variability, or the **natural tolerance** limits of a process
- **Specification** limits are determined externally, for example by customers or designers
- There is no mathematical or statistical relationship between the control limits and the specification limits

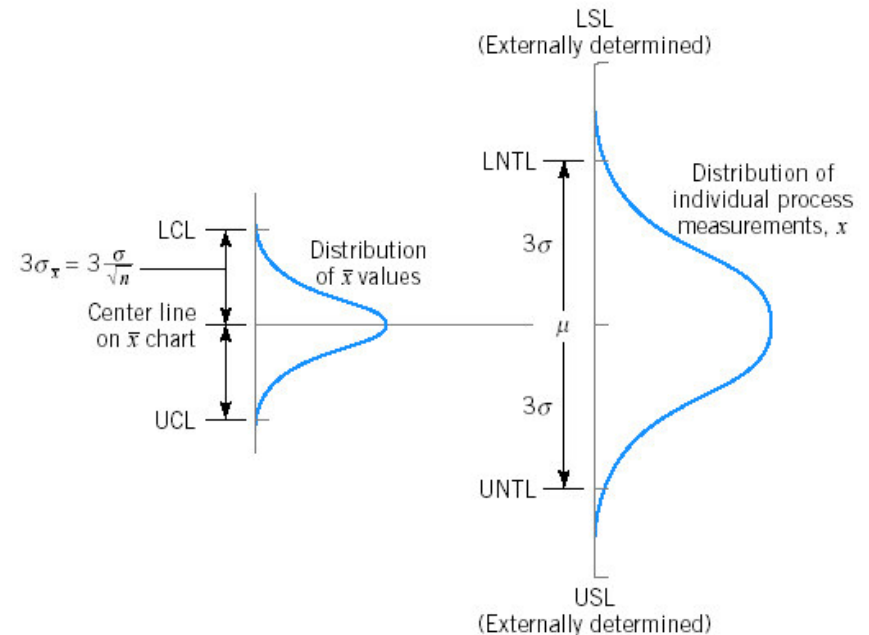


Figure 5-6 Relationship of natural tolerance limits, control limits, and specification limits.

Rational Subgroups

- $\bar{x}$ charts monitor **between-sample variability**
- R charts measure **within-sample variability**
- Standard deviation estimate of σ used to construct control limits is calculated from **within-sample variability**
- It is not correct to estimate σ using

$$s = \sqrt{\frac{\sum_{i=1}^m \sum_{j=1}^n (x_{ij} - \bar{\bar{x}})^2}{mn - 1}}$$

Guidelines for Control Chart Design

- Control chart design requires specification of sample size, control limit width, and sampling frequency
 - Exact solution requires detailed information on statistical characteristics as well as economic factors
 - The problem of choosing sample size and sampling frequency is one of **allocating sampling effort**
- For $\bar{X}$ chart, choose as small a sample size is consistent with magnitude of process shift one is trying to detect. For moderate to large shifts, relatively small samples are effective. For small shifts, larger samples are needed.
- For small samples, R chart is relatively insensitive to changes in process standard deviation. For larger samples ($n > 10$ or 12), s or s^2 charts are better choices.

Probability Limits on the $\bar{x}$ and R Charts

It is customary to express the control limits on the $\bar{x}$ and R charts as a multiple of the standard deviation of the statistic plotted on the charts. If the multiple chosen is k , then the limits are referred to as k -sigma limits, the usual choice being $k = 3$. As mentioned in Chapter 4, however, it is also possible to define the control limits by specifying the type I error level for the test. Such control limits are called probability limits and are used extensively in the United Kingdom and some parts of Western Europe.

It is easy to choose probability limits for the $\bar{x}$ chart. Since $\bar{x}$ is approximately normally distributed, we may obtain a desired type I error of α by choosing the multiple of sigma for the control limit as $k = Z_{\alpha/2}$, where $Z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution. Note that the usual three-sigma limits imply that the type I error probability is $\alpha = 0.0027$. If we choose $\alpha = 0.002$, for example, as most writers who recommend probability limits do, then $Z_{\alpha/2} = Z_{0.001} = 3.09$. Consequently, there is very little difference between such control limits and three-sigma control limits.

We may also construct R charts using probability limits. If $\alpha = 0.002$, the 0.001 and 0.999 percentage points of the distribution of the relative range $W = R/\sigma$ are required. These points obviously depend on the subgroup size n . Denoting these points by $W_{0.001}(n)$ and $W_{0.999}(n)$, and estimating σ by $\bar{R}/d_2$, we would have the 0.001 and 0.999 limits for R as $W_{0.001}(n)(\bar{R}/d_2)$ and $W_{0.999}(n)(\bar{R}/d_2)$. If we let $D_{0.001} = W_{0.001}(n)/d_2$ and $D_{0.999} = W_{0.999}(n)/d_2$, then the probability limits for the R chart are

$$\begin{aligned}\text{UCL} &= D_{0.999}\bar{R} \\ \text{LCL} &= D_{0.001}\bar{R}\end{aligned}$$

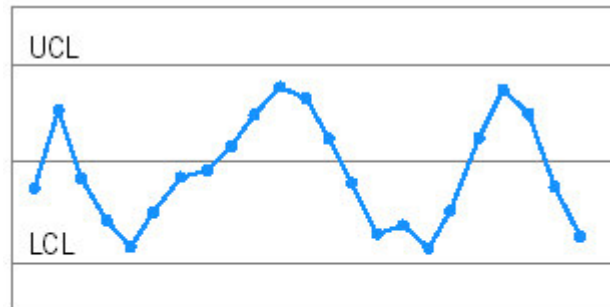
5-2.3 Charts Based on Standard Values

$$\begin{aligned} \text{UCL} &= \mu + 3 \frac{\sigma}{\sqrt{n}} \\ \text{Center line} &= \mu \\ \text{LCL} &= \mu - 3 \frac{\sigma}{\sqrt{n}} \end{aligned} \quad (5-14)$$

$$\begin{aligned} \text{UCL} &= \mu + A\sigma \\ \text{Center line} &= \mu \\ \text{LCL} &= \mu - A\sigma \end{aligned} \quad (5-15)$$

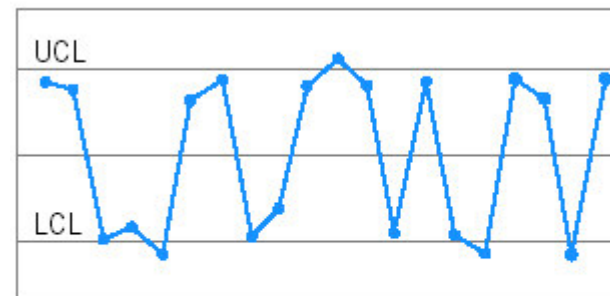
$$\begin{aligned} \text{UCL} &= D_2\sigma \\ \text{Center line} &= d_2\sigma \\ \text{LCL} &= D_1\sigma \end{aligned} \quad (5-17)$$

5-2.4 Interpretation of $\bar{x}$ and R Charts



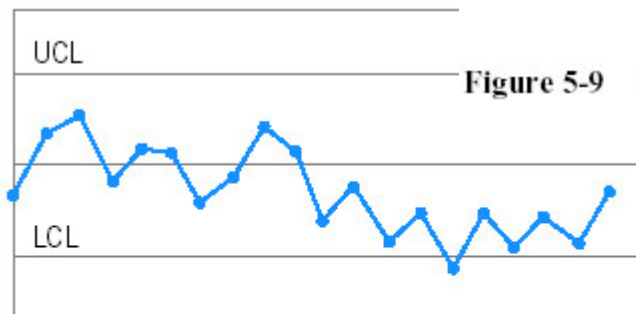
Sample number

Figure 5-8 Cycles on a control chart.



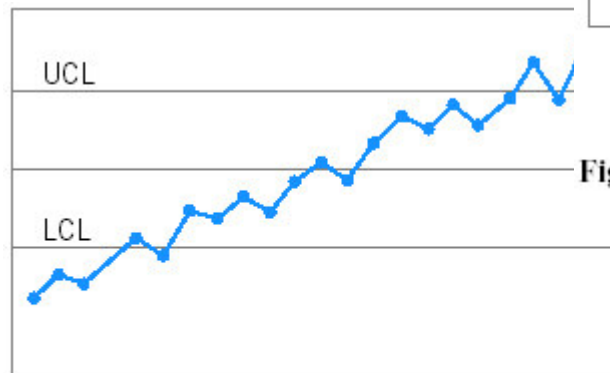
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Figure 5-9 A mixture pattern.



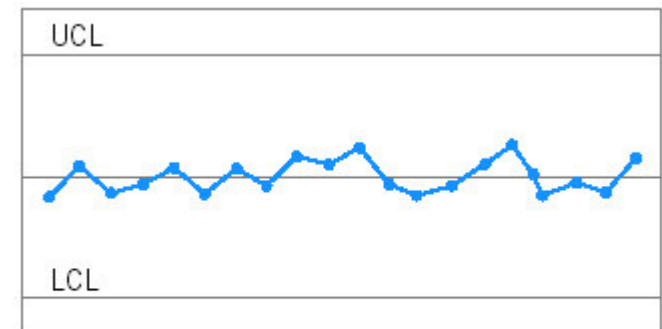
Sample number

Figure 5-10 A shift in process level.



Sample number

Figure 5-11 A trend in process level.



Sample number

Figure 5-12 A stratification pattern.

5-2.5 The Effect of Nonnormality on $\bar{x}$ and R Charts

- An assumption in performance properties is that the underlying distribution of quality characteristic is **normal**
 - If underlying distribution is not normal, sampling distributions can be derived and exact probability limits obtained
- Burr (1967) notes the usual normal theory control limits are very robust to normality assumption
- Schilling and Nelson (1976) indicate that in most cases, samples of size 4 or 5 are sufficient to ensure reasonable robustness to normality assumption for $\bar{x}$ chart
- Sampling distribution of R is **not** symmetric, thus symmetric 3-sigma limits are an approximation and α -risk is not 0.0027. R chart is more sensitive to departures from normality than $\bar{x}$ chart.
- Assumptions of normality and independence are not a primary concern in phase I

5-2.6 The Operating-Characteristic Function

$$\beta = \Phi(L - k\sqrt{n}) - \Phi(-L - k\sqrt{n}) \quad (5-19)$$

To illustrate the use of equation 5-19, suppose that we are using an $\bar{x}$ chart with $L = 3$ (the usual three-sigma limits), the sample size $n = 5$, and we wish to determine the probability of detecting a shift to $\mu_1 = \mu_0 + 2\sigma$ on the first sample following the shift. Then, since $L = 3$, $k = 2$, and $n = 5$, we have

$$\begin{aligned} \beta &= \Phi[3 - 2\sqrt{5}] - \Phi[-3 - 2\sqrt{5}] \\ &= \Phi(-1.47) - \Phi(-7.37) \\ &\cong 0.0708 \end{aligned}$$

In general, the expected number of samples taken before the shift is detected is simply the **average run length**, or

$$ARL = \sum_{r=1}^{\infty} r\beta^{r-1}(1-\beta) = \frac{1}{1-\beta}$$

Therefore, in our example, we have

$$ARL = \frac{1}{1-\beta} = \frac{1}{0.25} = 4$$

In other words, the expected number of samples taken to detect a shift of 1.0σ with $n = 5$ is 4.

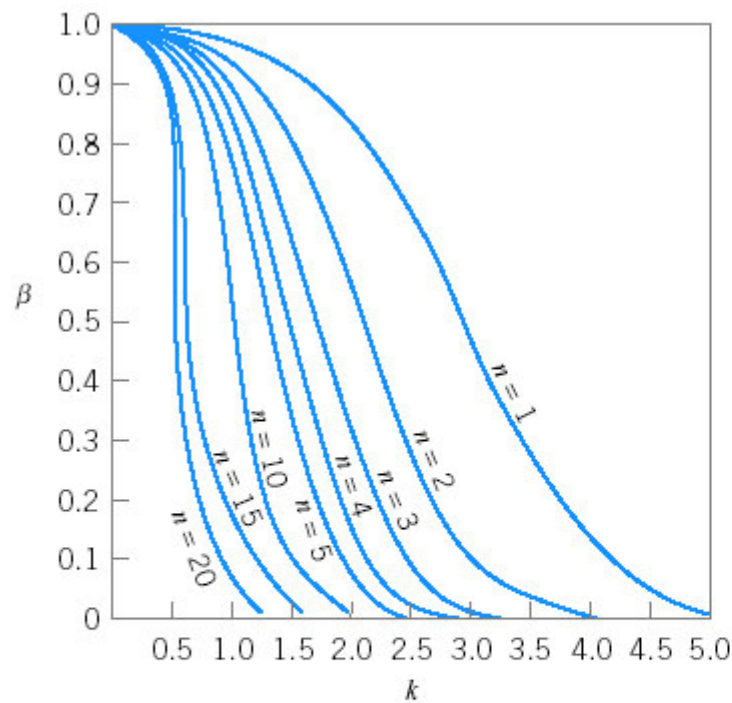


Figure 5-13 Operating-characteristic curves for the $\bar{x}$ chart with three-sigma limits. $\beta = P$ (not detecting a shift of $k\sigma$ in the mean on the first sample following the shift).

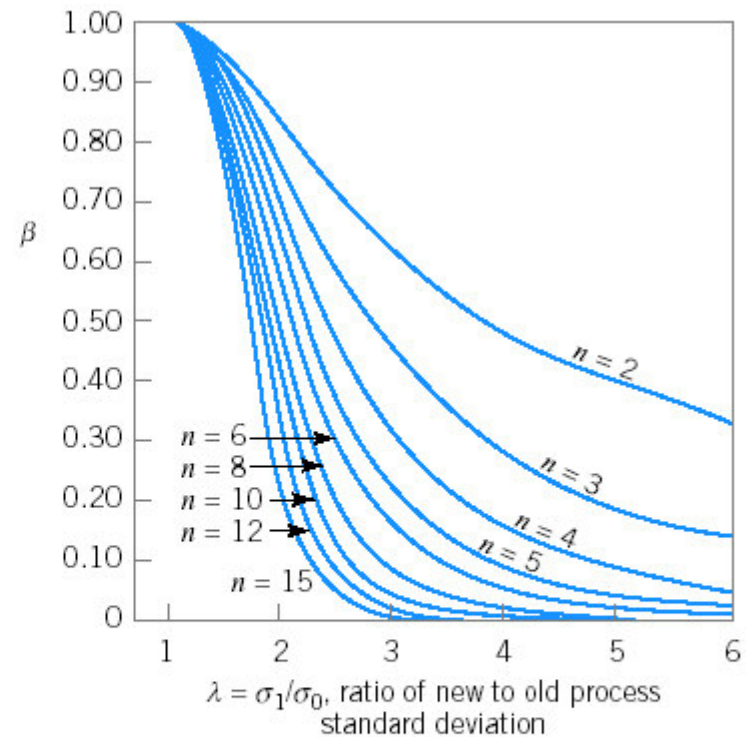


Figure 5-14 Operating-characteristic curves for the R chart with three-sigma limits. (Adapted from A. J. Duncan, "Operating Characteristics of R Charts," *Industrial Quality Control*, vol. 7, no. 5, pp. 40–41, 1951, with permission of the American Society for Quality Control.)

5-2.7 The Average Run Length for the $\bar{x}$ Chart

For any Shewhart control chart, we have noted previously that the ARL can be expressed as

$$ARL = \frac{1}{P(\text{one point plots out of control})}$$

or

$$ARL_0 = \frac{1}{\alpha} \quad (5-20)$$

for the in-control ARL and

$$ARL_1 = \frac{1}{1 - \beta} \quad (5-21)$$

$$ATS = ARL \, h \quad (5-22)$$

$$I = n \, ARL \quad (5-23)$$

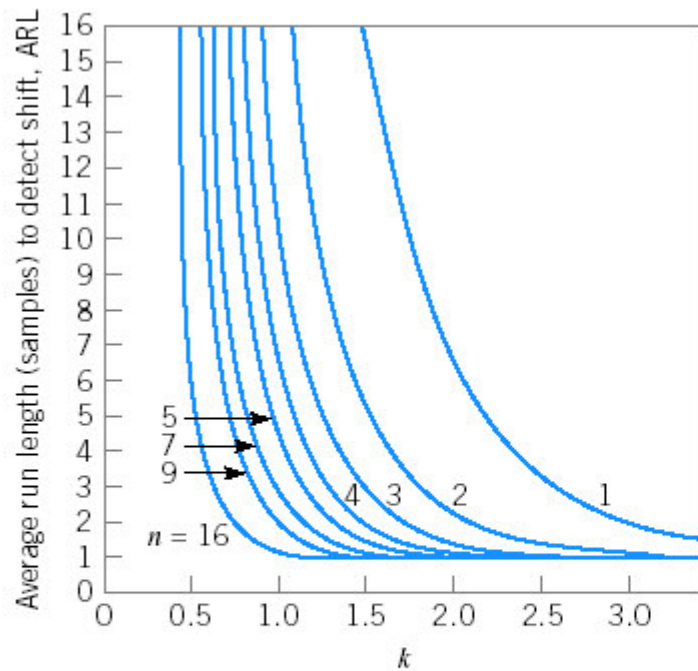


Figure 5-15 Average run length (samples) for the $\bar{x}$ chart with 3-sigma limits, where the process mean shifts by $k\sigma$. (Adapted from *Modern Methods for Quality Control and Improvement*, by H. M. Wadsworth, K. S. Stephens, and A. B. Godfrey, 2nd edition, John Wiley & Sons, 2002.)

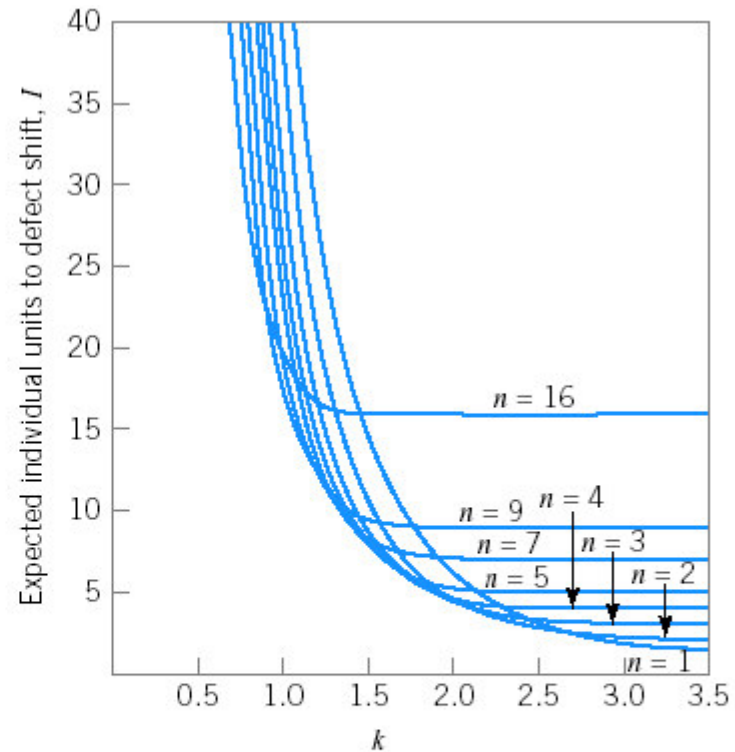


Figure 5-16 Average run length (individual units) for the $\bar{x}$ chart with three-sigma limits, where the process mean shifts by $k\sigma$. (Adapted from *Modern Methods for Quality Control and Improvement*, by H. M. Wadsworth, K. S. Stephens, and A. B. Godfrey, 2nd edition, John Wiley & Sons, 2002.)

5-3 CONTROL CHARTS FOR $\bar{x}$ AND s

Although $\bar{x}$ and R charts are widely used, it is occasionally desirable to estimate the process standard deviation directly instead of indirectly through the use of the range R . This leads to control charts for $\bar{x}$ and s , where s is the sample standard deviation.¹ Generally, $\bar{x}$ and s charts are preferable to their more familiar counterparts, $\bar{x}$ and R charts, when either

1. The sample size n is moderately large—say, $n > 10$ or 12. (Recall that the range method for estimating σ loses statistical efficiency for moderate to large samples.)
2. The sample size n is variable.

In this section, we illustrate the construction and operation of $\bar{x}$ and s control charts. We also show how to deal with variable sample size and discuss an alternative to the s chart.

$$\begin{aligned} \text{UCL} &= B_6\sigma \\ \text{Center line} &= c_4\sigma \\ \text{LCL} &= B_5\sigma \end{aligned} \tag{5-25}$$

If no standard is given for σ , then it must be estimated by analyzing past data. Suppose that m preliminary samples are available, each of size n , and let s_i be the standard deviation of the i th sample. The average of the m standard deviations is

$$\bar{s} = \frac{1}{m} \sum_{i=1}^m s_i$$

The statistic $\bar{S}/c_4$ is an unbiased estimator of σ . Therefore, the parameters of the s chart would be

$$\text{UCL} = \bar{s} + 3 \frac{\bar{s}}{c_4} \sqrt{1 - c_4^2}$$

$$\text{Center line} = \bar{s}$$

$$\text{LCL} = \bar{s} - 3 \frac{\bar{s}}{c_4} \sqrt{1 - c_4^2}$$

We usually define the constants

$$B_3 = 1 - \frac{3}{c_4} \sqrt{1 - c_4^2} \quad \text{and} \quad B_4 = 1 + \frac{3}{c_4} \sqrt{1 - c_4^2} \quad (5-26)$$

Consequently, we may write the parameters of the s chart as

$$\begin{aligned} \text{UCL} &= B_4 \bar{s} \\ \text{Center line} &= \bar{s} \\ \text{LCL} &= B_3 \bar{s} \end{aligned} \quad (5-27)$$

When $\bar{s}/c_4$ is used to estimate σ , we may define the control limits on the corresponding $\bar{x}$ chart as

$$\text{UCL} = \bar{\bar{x}} + \frac{3\bar{s}}{c_4\sqrt{n}}$$

$$\text{Center line} = \bar{\bar{x}}$$

$$\text{LCL} = \bar{\bar{x}} - \frac{3\bar{s}}{c_4\sqrt{n}}$$

Let the constant $A_3 = 3/(c_4\sqrt{n})$. Then the $\bar{x}$ chart parameters become

$$\text{UCL} = \bar{\bar{x}} + A_3\bar{s}$$

$$\text{Center line} = \bar{\bar{x}}$$

$$\text{LCL} = \bar{\bar{x}} - A_3\bar{s}$$

(5-28)

EXAMPLE 5-3

Table 5-3 Inside Diameter Measurements (mm) for Automobile Engine Piston Rings

Sample Number	Observations					$\bar{x}_i$	S_i
1	74.030	74.002	74.019	73.992	74.008	74.010	0.0148
2	73.995	73.992	74.001	74.011	74.004	74.001	0.0075
3	73.988	74.024	74.021	74.005	74.002	74.008	0.0147
4	74.002	73.996	73.993	74.015	74.009	74.003	0.0091
5	73.992	74.007	74.015	73.989	74.014	74.003	0.0122
6	74.009	73.994	73.997	73.985	73.993	73.996	0.0087
7	73.995	74.006	73.994	74.000	74.005	74.000	0.0055
8	73.985	74.003	73.993	74.015	73.988	73.997	0.0123
9	74.008	73.995	74.009	74.005	74.004	74.004	0.0055
10	73.998	74.000	73.990	74.007	73.995	73.998	0.0063
11	73.994	73.998	73.994	73.995	73.990	73.994	0.0029
12	74.004	74.000	74.007	74.000	73.996	74.001	0.0042
13	73.983	74.002	73.998	73.997	74.012	73.998	0.0105
14	74.006	73.967	73.994	74.000	73.984	73.990	0.0153
15	74.012	74.014	73.998	73.999	74.007	74.006	0.0073
16	74.000	73.984	74.005	73.998	73.996	73.997	0.0078
17	73.994	74.012	73.986	74.005	74.007	74.001	0.0106
18	74.006	74.010	74.018	74.003	74.000	74.007	0.0070
19	73.984	74.002	74.003	74.005	73.997	73.998	0.0085
20	74.000	74.010	74.013	74.020	74.003	74.009	0.0080
21	73.982	74.001	74.015	74.005	73.996	74.000	0.0122
22	74.004	73.999	73.990	74.006	74.009	74.002	0.0074
23	74.010	73.989	73.990	74.009	74.014	74.002	0.0119
24	74.015	74.008	73.993	74.000	74.010	74.005	0.0087
25	73.982	73.984	73.995	74.017	74.013	73.998	0.0162
						$\Sigma = 1850.028$	0.2351
						$\bar{x} = 74.001$	$\bar{s} = 0.0094$

We will illustrate the construction of $\bar{x}$ and s charts using the piston-ring inside diameter measurements in Table 5-3. The grand average and the average standard deviation are

$$\bar{\bar{x}} = \frac{1}{25} \sum_{i=1}^{25} \bar{x}_i = \frac{1}{25}(1850.028) = 74.001$$

and

$$\bar{s} = \frac{1}{25} \sum_{i=1}^{25} s_i = \frac{1}{25}(0.2351) = 0.0094$$

respectively. Consequently, the parameters for the $\bar{x}$ chart are

$$UCL = \bar{\bar{x}} + A_3\bar{s} = 74.001 + (1.427)(0.0094) = 74.014$$

$$CL = \bar{\bar{x}} = 74.001$$

$$LCL = \bar{\bar{x}} - A_3\bar{s} = 74.001 - (1.427)(0.0094) = 73.988$$

and for the s chart

$$UCL = B_4\bar{s} = (2.089)(0.0094) = 0.0196$$

$$CL = \bar{s} = 0.0094$$

$$LCL = B_3\bar{s} = (0)(0.0094) = 0$$

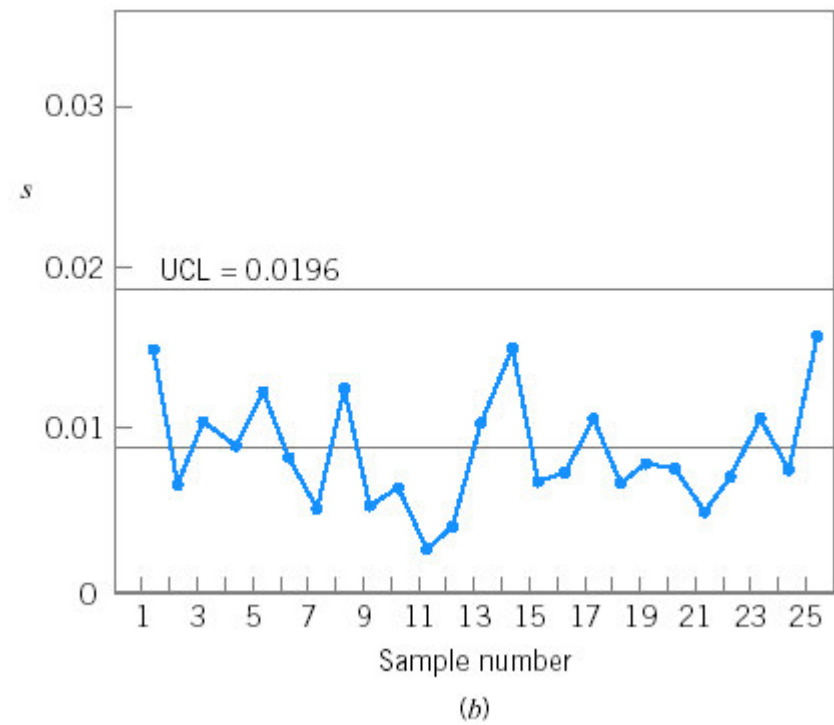
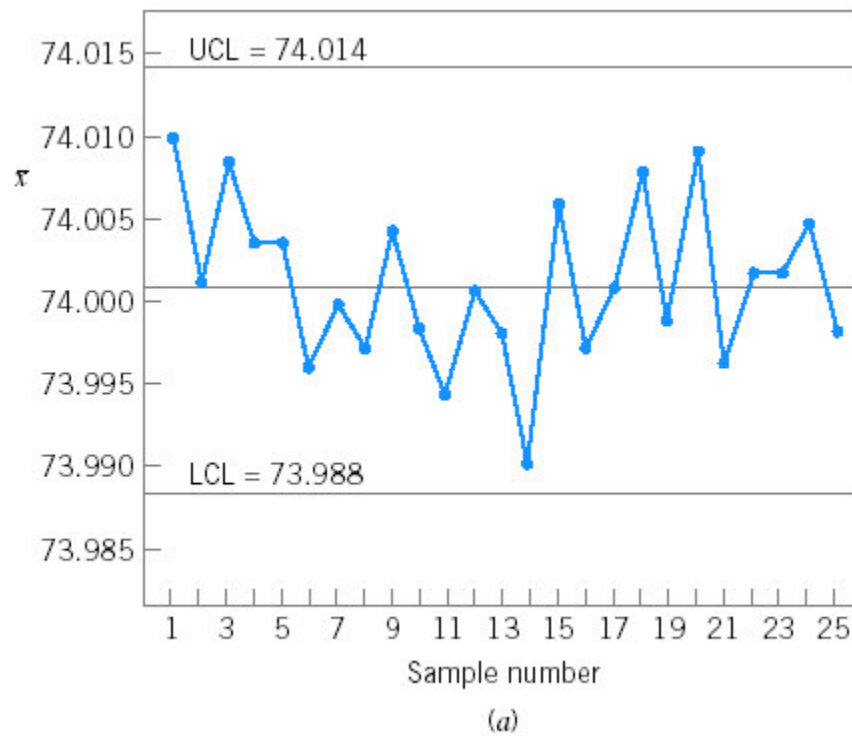


Figure 5-17 The $\bar{x}$ and s control charts for Example 5-3. (a) The $\bar{x}$ chart with control limits based on $\bar{s}$. (b) The s control chart.

5-3.2 The $\bar{x}$ and s Control Charts with Variable Sample Size

$$\bar{\bar{x}} = \frac{\sum_{i=1}^m n_i \bar{x}_i}{\sum_{i=1}^m n_i} \quad (5-30)$$

and

$$\bar{s} = \left[\frac{\sum_{i=1}^m (n_i - 1) s_i^2}{\sum_{i=1}^m n_i - m} \right]^{1/2} \quad (5-31)$$

EXAMPLE 5-4

Table 5-4 Inside Diameter Measurements (mm) on Automobile Engine Piston Rings

Sample Number	Observations					$\bar{x}_i$	s_i
1	74.030	74.002	74.019	73.992	74.008	74.010	0.0148
2	73.995	73.992	74.001			73.996	0.0046
3	73.988	74.024	74.021	74.005	74.002	74.008	0.0147
4	74.002	73.996	73.993	74.015	74.009	74.003	0.0091
5	73.992	74.007	74.015	73.989	74.014	74.003	0.0122
6	74.009	73.994	73.997	73.985		73.996	0.0099
7	73.995	74.006	73.994	74.000		73.999	0.0055
8	73.985	74.003	73.993	74.015	73.988	73.997	0.0123
9	74.008	73.995	74.009	74.005		74.004	0.0064
10	73.998	74.000	73.990	74.007	73.995	73.998	0.0063
11	73.994	73.998	73.994	73.995	73.990	73.994	0.0029
12	74.004	74.000	74.007	74.000	73.996	74.001	0.0042
13	73.983	74.002	73.998			73.994	0.0100
14	74.006	73.967	73.994	74.000	73.984	73.990	0.0153
15	74.012	74.014	73.998			74.008	0.0087
16	74.000	73.984	74.005	73.998	73.996	73.997	0.0078
17	73.994	74.012	73.986	74.005		73.999	0.0115
18	74.006	74.010	74.018	74.003	74.000	74.007	0.0070
19	73.984	74.002	74.003	74.005	73.997	73.998	0.0085
20	74.000	74.010	74.013			74.008	0.0068
21	73.982	74.001	74.015	74.005	73.996	74.000	0.0122
22	74.004	73.999	73.990	74.006	74.009	74.002	0.0074
23	74.010	73.989	73.990	74.009	74.014	74.002	0.0119
24	74.015	74.008	73.993	74.000	74.010	74.005	0.0087
25	73.982	73.984	73.995	74.017	74.013	73.998	0.0162

Consider the data in Table 5-4, which is a modification of the piston-ring data used in Example 5-3. Note that the sample sizes vary from $n = 3$ to $n = 5$. We may use the procedure described above to set up the $\bar{x}$ and s control charts. The weighted grand mean and weighted average standard deviation are computed from equations 5-30 and 5-31 as follows:

$$\begin{aligned}\bar{\bar{x}} &= \frac{\sum_{i=1}^{25} n_i \bar{x}_i}{\sum_{i=1}^{25} n_i} = \frac{5(74.010) + 3(73.996) + \cdots + 5(73.998)}{5 + 3 + \cdots + 5} \\ &= \frac{8362.075}{113} = 74.001\end{aligned}$$

and

$$\begin{aligned}\bar{s} &= \left[\frac{\sum_{i=1}^{25} (n_i - 1) s_i^2}{\sum_{i=1}^{25} n_i - 25} \right]^{1/2} = \left[\frac{4(0.0148)^2 + 2(0.0046)^2 + \cdots + 4(0.0162)^2}{5 + 3 + \cdots + 5 - 25} \right]^{1/2} \\ &= \left[\frac{0.009324}{88} \right]^{1/2} = 0.0103\end{aligned}$$

Therefore, the center line of the $\bar{x}$ chart is $\bar{\bar{x}} = 74.001$, and the center line of the s chart is $\bar{s} = 0.0103$. The control limits may now be easily calculated. To illustrate, consider the first sample. The limits for the $\bar{x}$ chart are

$$\text{UCL} = 74.001 + (1.427)(0.0103) = 74.016$$

$$\text{CL} = 74.001$$

$$\text{LCL} = 74.001 - (1.427)(0.0103) = 73.986$$

The control limits for the s chart are

$$UCL = (2.089)(0.0103) = 0.022$$

$$CL = 0.0098$$

$$LCL = 0(0.0103) = 0$$

Note that we have used the values of A_3 , B_3 , and B_4 for $n_1 = 5$. The limits for the second sample would use the values of these constants for $n_2 = 3$. The control limit calculations for all 25 samples are summarized in Table 5-5. The control charts are plotted in Fig. 5-18.

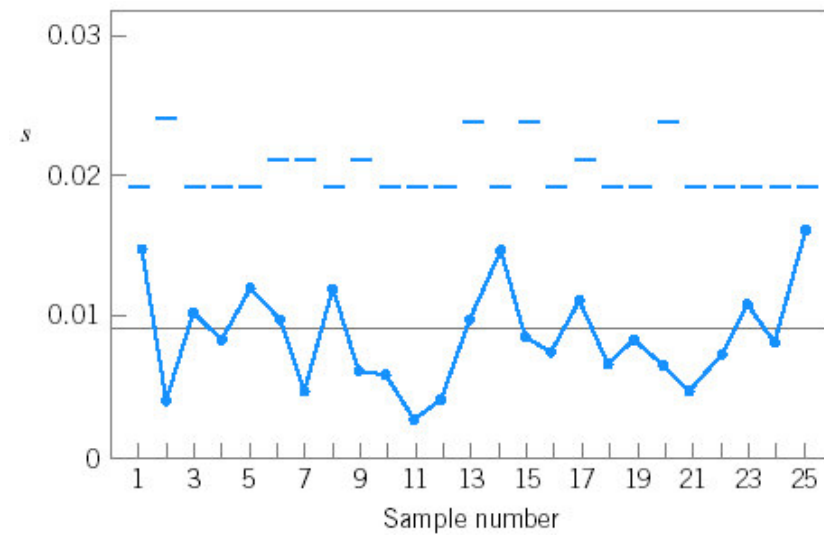
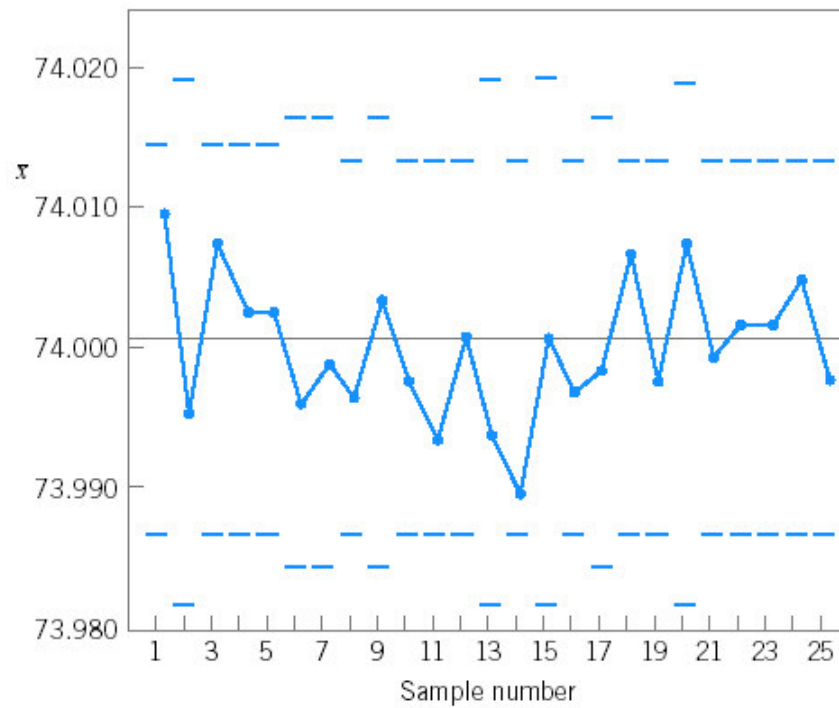


Figure 5-18 The $\bar{x}$ and s control charts for piston-ring data with variable sample size, Example 5-4.

5-3.3 The s^2 Control Chart

Most quality engineers use either the R chart or the s chart to monitor process variability, with s preferable to R for moderate to large sample sizes. Some practitioners recommend a control chart based directly on the sample variance s^2 . The parameters for the s^2 control chart are

$$\begin{aligned} \text{UCL} &= \frac{\bar{s}^2}{n-1} \chi_{\alpha/2, n-1}^2 \\ \text{Center line} &= \bar{s}^2 \\ \text{LCL} &= \frac{\bar{s}^2}{n-1} \chi_{1-(\alpha/2), n-1}^2 \end{aligned} \quad (5-32)$$

5-4 THE SHEWHART CONTROL CHART FOR INDIVIDUAL MEASUREMENTS

There are many situations in which the sample size used for process monitoring is $n = 1$; that is, the sample consists of an individual unit. Some examples of these situations are as follows:

1. Automated inspection and measurement technology is used, and every unit manufactured is analyzed so there is no basis for rational subgrouping.
2. The production rate is very slow, and it is inconvenient to allow sample sizes of $n > 1$ to accumulate before analysis. The long interval between observations will cause problems with rational subgrouping.
3. Repeat measurements on the process differ only because of laboratory or analysis error, as in many chemical processes.
4. Multiple measurements are taken on the same unit of product, such as measuring oxide thickness at several different locations on a wafer in semiconductor manufacturing.
5. In process plants, such as papermaking, measurements on some parameter such as coating thickness *across* the roll will differ very little and produce a standard deviation that is much too small if the objective is to control coating thickness *along* the roll.

In many applications of the individuals control chart we use the moving range of two successive observations as the basis of estimating the process variability. The moving range is defined as

$$MR_i = |x_i - x_{i-1}|$$

EXAMPLE 5-5

Table 5-6 Viscosity of Aircraft Primer Paint

Batch Number	Viscosity x	Moving Range MR
1	34.05	
2	34.40	0.35
3	33.59	0.81
4	35.96	2.37
5	34.70	1.26
6	33.51	1.19
7	33.79	0.28
8	34.04	0.25
9	34.52	0.48
10	33.75	0.77
11	33.27	0.48
12	33.71	0.44
13	34.03	0.32
14	34.58	0.55
15	34.02	0.56
16	33.97	0.05
17	34.05	0.08
18	34.04	0.01
19	33.73	0.31
20	34.05	0.32
	$\bar{x} = 34.088$	$\overline{MR} = 0.5726$

The viscosity of an aircraft primer paint is an important quality characteristic. The product is produced in batches, and because each batch takes several hours to produce, the production rate is too slow to allow for rational subgroups of size greater than one. The viscosity of the previous 20 batches is shown in Table 5-6.

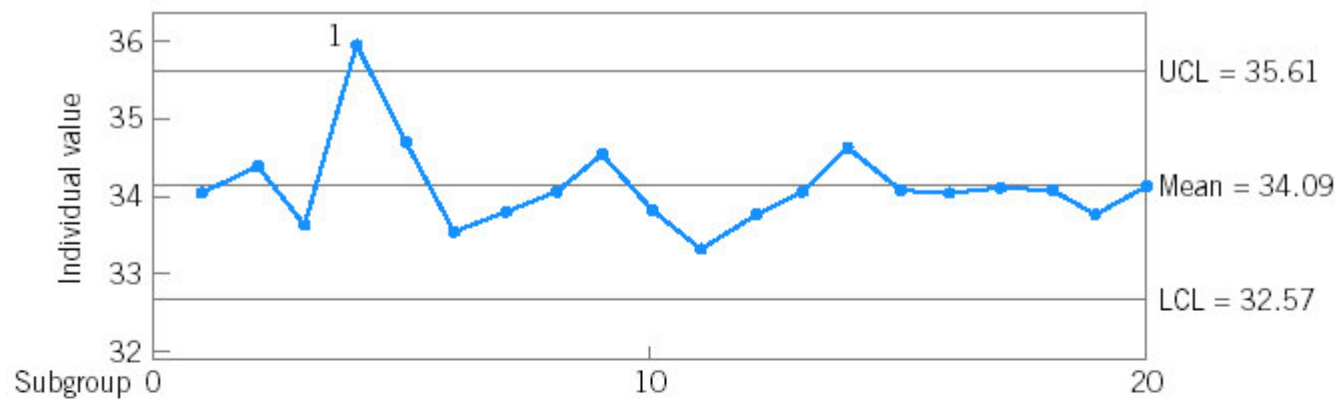
To set up the control chart for individual observations, note that the sample average of the 20 viscosity readings is $\bar{x} = 34.088$ and that the average of the moving ranges of two observations is $\overline{MR} = 0.573$. To set up the moving-range chart, we use $D_3 = 0$ and $D_4 = 3.267$ for $n = 2$. Therefore, the moving-range chart has center line $\overline{MR} = 0.5726$, LCL = 0, and $UCL = D_4\overline{MR} = (3.267)(0.5726) = 1.871$. The control chart (from Minitab) is shown in Fig. 5-19b. Notice that the point for sample 4 is out of control.

For the control chart for individual measurements, the parameters are

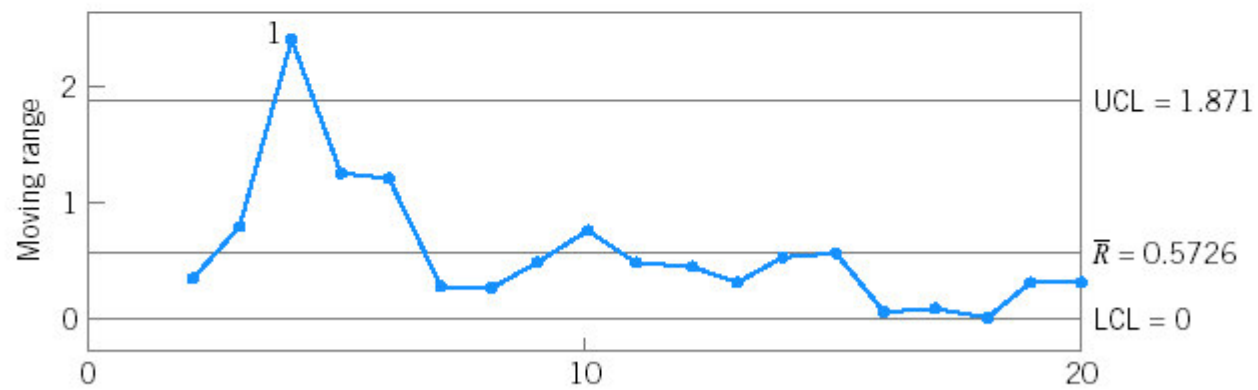
$$\begin{aligned} UCL &= \bar{x} + 3 \frac{\overline{MR}}{d_2} \\ \text{Center line} &= \bar{x} \\ LCL &= \bar{x} - 3 \frac{\overline{MR}}{d_2} \end{aligned} \quad (5-33)$$

If a moving range of $n = 2$ observations is used, then $d_2 = 1.128$. For the data in Table 5-6, we have

$$\begin{aligned} UCL &= \bar{x} + 3 \frac{\overline{MR}}{d_2} = 34.088 + 3 \frac{0.5726}{1.128} = 35.61 \\ \text{Center line} &= \bar{x} = 34.088 \\ LCL &= \bar{x} - 3 \frac{\overline{MR}}{d_2} = 34.088 - 3 \frac{0.5726}{1.128} = 32.57 \end{aligned}$$



(a)



(b)

Figure 5-19 Control charts for (a) the moving range and for (b) individual observations on viscosity.

Phase II Operation and Interpretation of the Charts

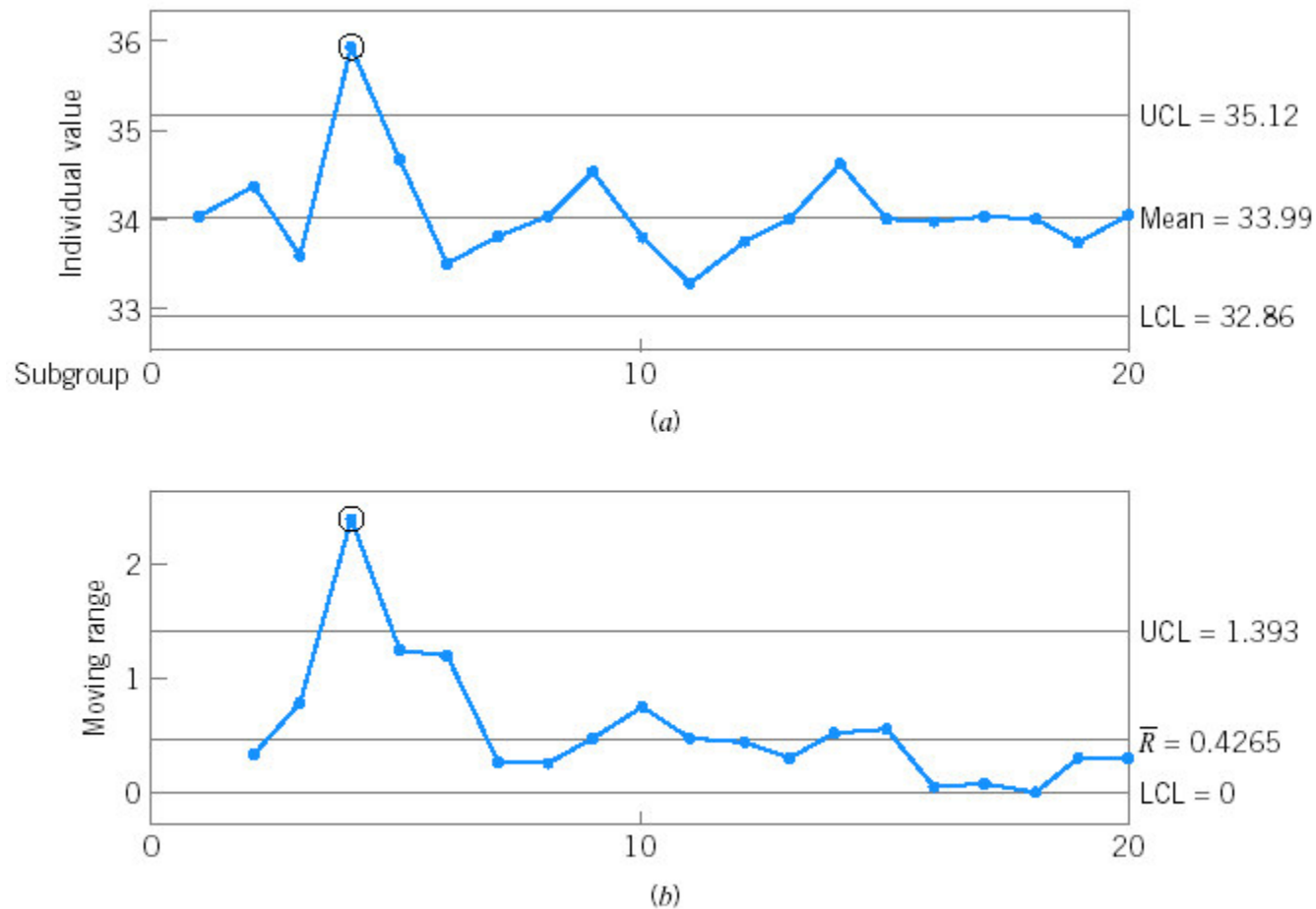


Figure 5-20 Revised individuals and moving-range control charts for viscosity.

Table 5-7 Viscosity for Aircraft Primer Paint, Batches 16-30

Batch Number	Viscosity x	Moving Range MR
21 ^a	34.39	0.34
22	33.75	0.64
23	33.40	0.35
24	33.27	0.13
25	34.65	1.38
26	34.80	0.15
27	34.55	0.25
28	35.40	0.85
29	34.75	0.65
30	34.50	0.25
31	34.70	0.20
32	34.29	0.41
33	34.61	0.32
34	34.49	0.12
35	35.03	0.54

^aThe moving range for batch 21 was computed as the difference in viscosity readings between batches 21 and 20; that is, $MR_{21} = |x_{21} - x_{20}| = |(34.39 - 34.05)| = 0.34$.

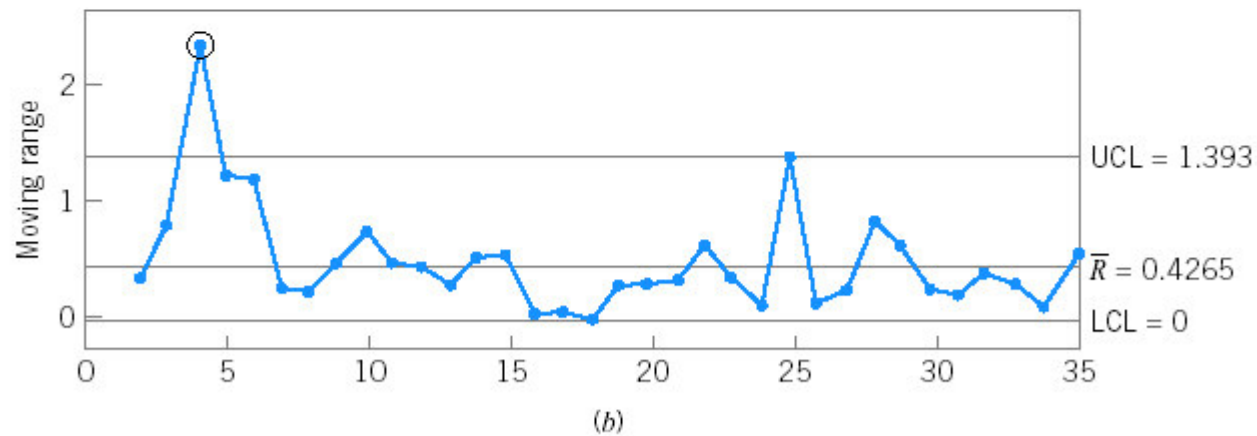
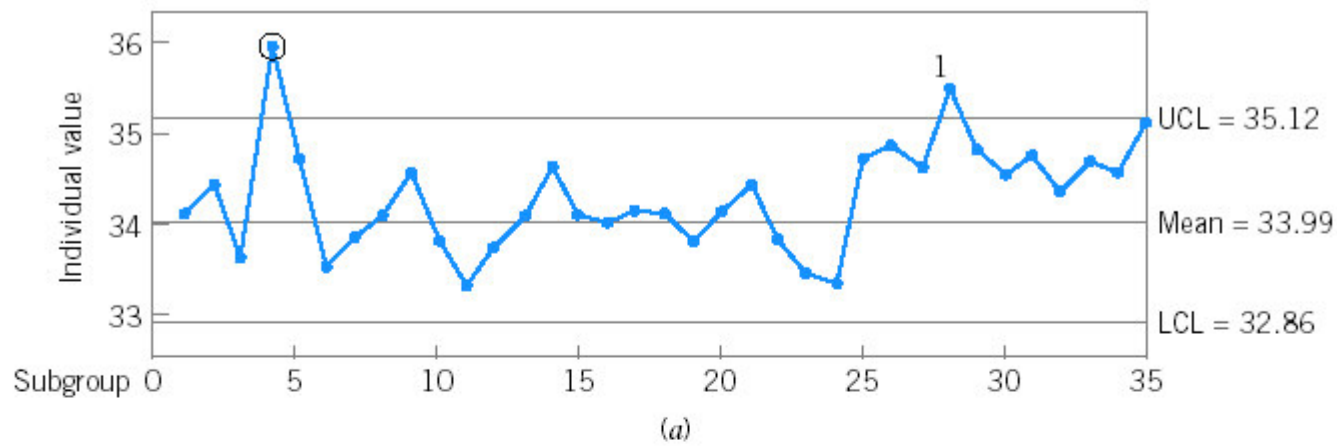


Figure 5-21 Continuation of the control chart for individuals and the moving range using the additional data in Table 5-7.

Average Run Lengths

- Crowder (1987b) showed that ARL_0 of combined individuals and moving-range chart with conventional 3-sigma limits is generally much less than ARL_0 ($= 370$) of standard Shewhart control chart
- Ability of individuals chart to detect small shifts is very poor
 - Rather than narrowing the 3-sigma limits, correct approach to detecting small shifts is a cumulative-sum or exponentially weighted moving-average control chart (Chapter 8)

Size of Shift	β	ARL_1
1σ	0.9772	43.96
2σ	0.8413	6.30
3σ	0.5000	2.00

Normality

- Borrer, Montgomery, and Runger (1999) found in-control ARL is dramatically affected by nonnormal data
- One approach for nonnormal data is to determine control limits for individuals control chart based on percentiles of correct underlying distribution
 - Requires at least 100 and preferably 200 observations

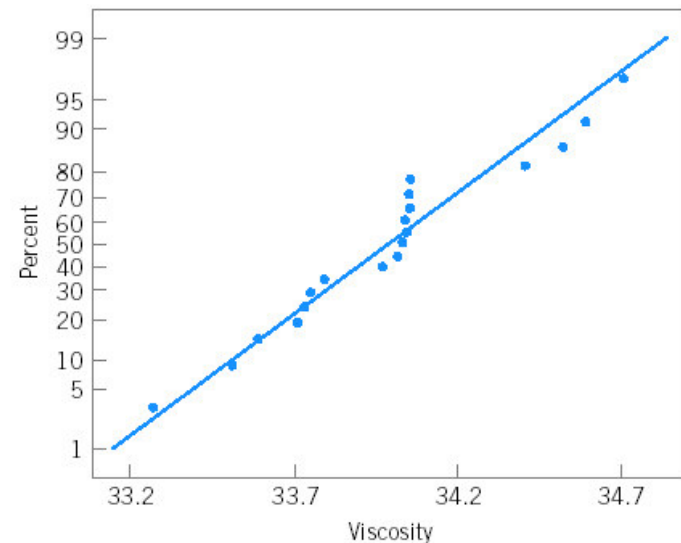


Figure 5-22 Normal probability plot of the primer paint viscosity data from Table 5-6, Example 5-5.

EXAMPLE 5-6

Table 5-8 Resistivity Data for Example 5-6

Sample, i	Resistivity (x_i)	$\ln(x_i)$	MR
1	216	5.37528	
2	290	5.66988	0.29460
3	236	5.46383	0.20605
4	228	5.42935	0.03448
5	244	5.49717	0.06782
6	210	5.34711	0.15006
7	139	4.93447	0.41264
8	310	5.73657	0.80210
9	240	5.48064	0.25593
10	211	5.35186	0.12878
11	175	5.16479	0.18707
12	447	6.10256	0.93777
13	307	5.72685	0.37571
14	242	5.48894	0.23791
15	168	5.12396	0.36498
16	360	5.88610	0.76214
17	226	5.42053	0.46557
18	253	5.53339	0.11286
19	380	5.94017	0.40678
20	131	4.87520	1.06497
21	173	5.15329	0.27809
22	224	5.41165	0.25836
23	195	5.27300	0.13865
24	199	5.29330	0.02030
25	226	5.42053	0.12723
		$\overline{\ln(x_i)} = 5.44402$	$\overline{MR} = 0.33712$

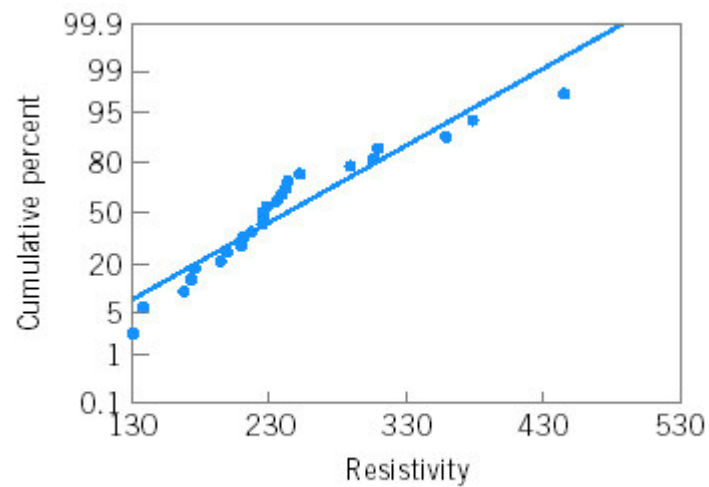


Figure 5-23 Normal probability plot of resistivity.

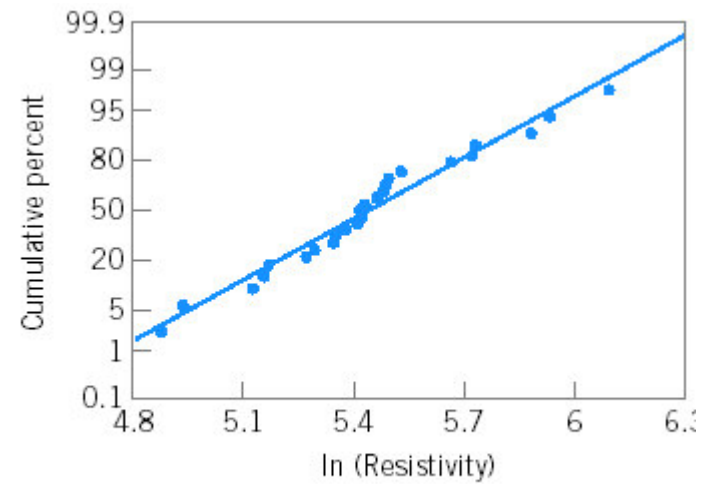


Figure 5-24 Normal probability plot of $\ln(\text{resistivity})$.

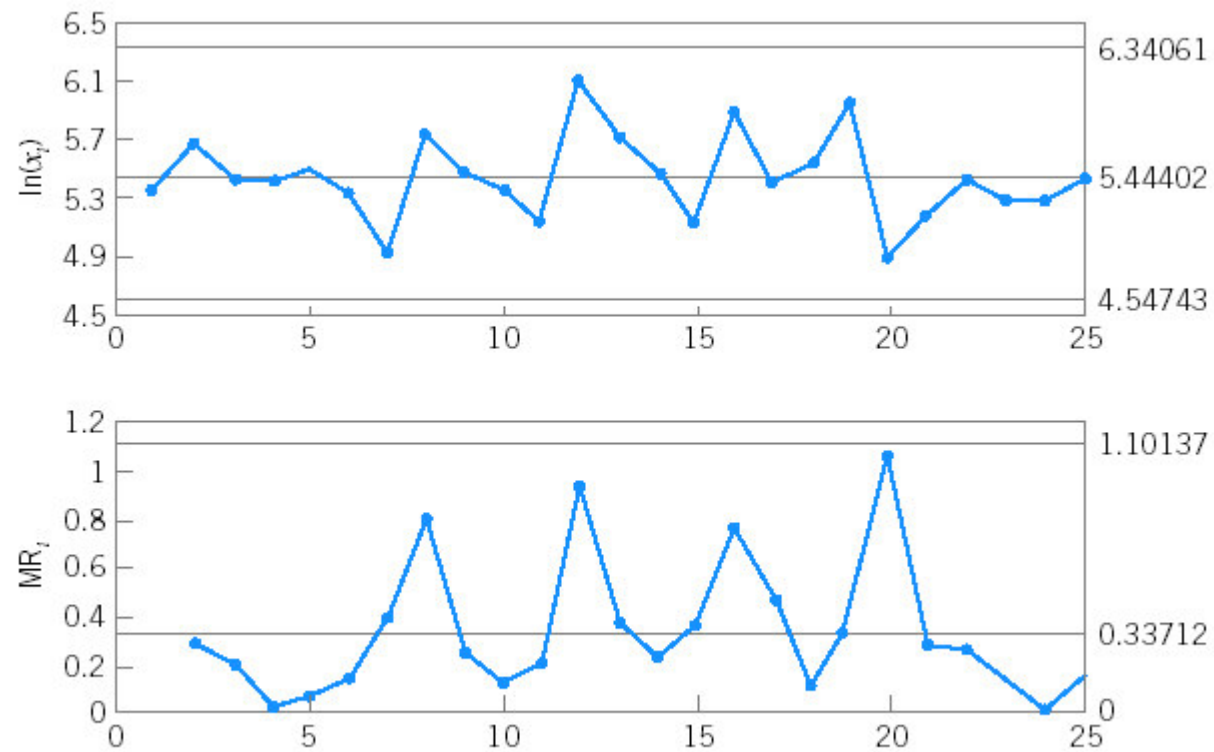


Figure 5-25 Individuals and moving range control charts on $\ln(\text{resistivity})$, Example 5-6.

EXAMPLE 5-11

The Need for Care in Selecting Rational Subgroups

Figure 5-26a shows a casting used in a gas turbine jet aircraft engine. This part is typical of those produced by both casting and machining processes for use in gas turbine engines and auxiliary power units in the aerospace industry—cylindrical parts created by rotating the cross section around a central axis. The vane height on this part is a critical quality characteristic.

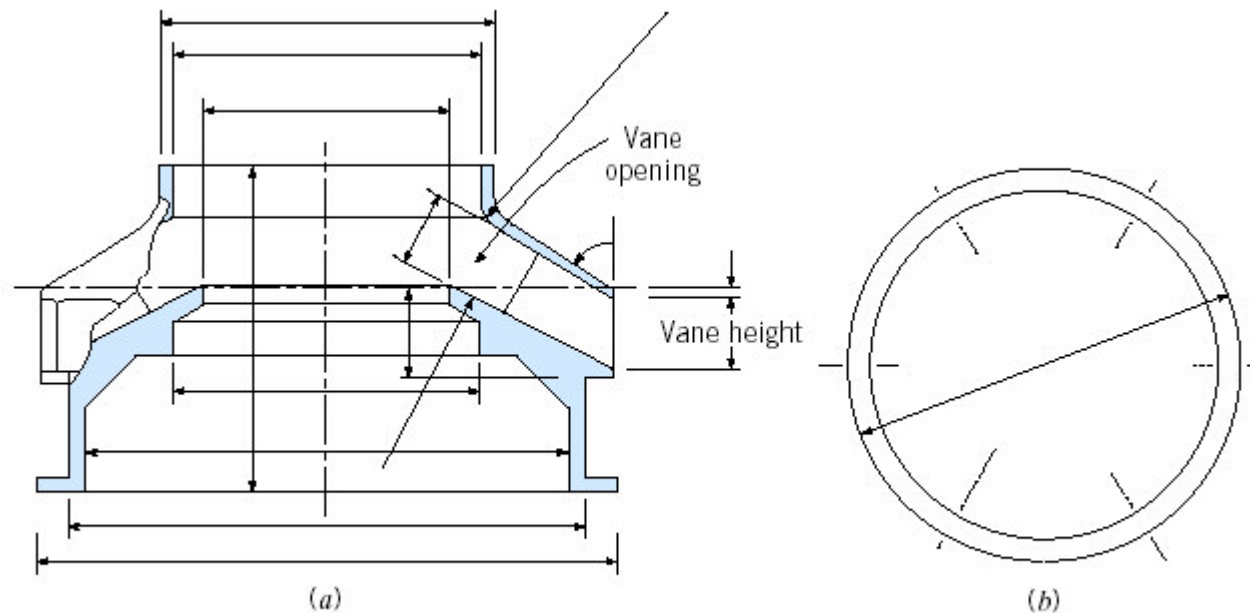


Figure 5-26 An aerospace casting.

Data on vane heights are collected by randomly selecting five vanes on each casting produced. Initially, the company constructed $\bar{x}$ and s control charts on these data to control and improve the process. This usually produced many out-of-control points on the $\bar{x}$ chart, with an occasional out-of-control point on the s chart. Figure 5-27 shows typical $\bar{x}$ and s charts for 20 castings. A more careful analysis of the control-charting procedure revealed that the chief problem was the use of the five measurements on a single part as a rational subgroup, and that the out-of-control conditions on the $\bar{x}$ chart did not provide a valid basis for corrective action.

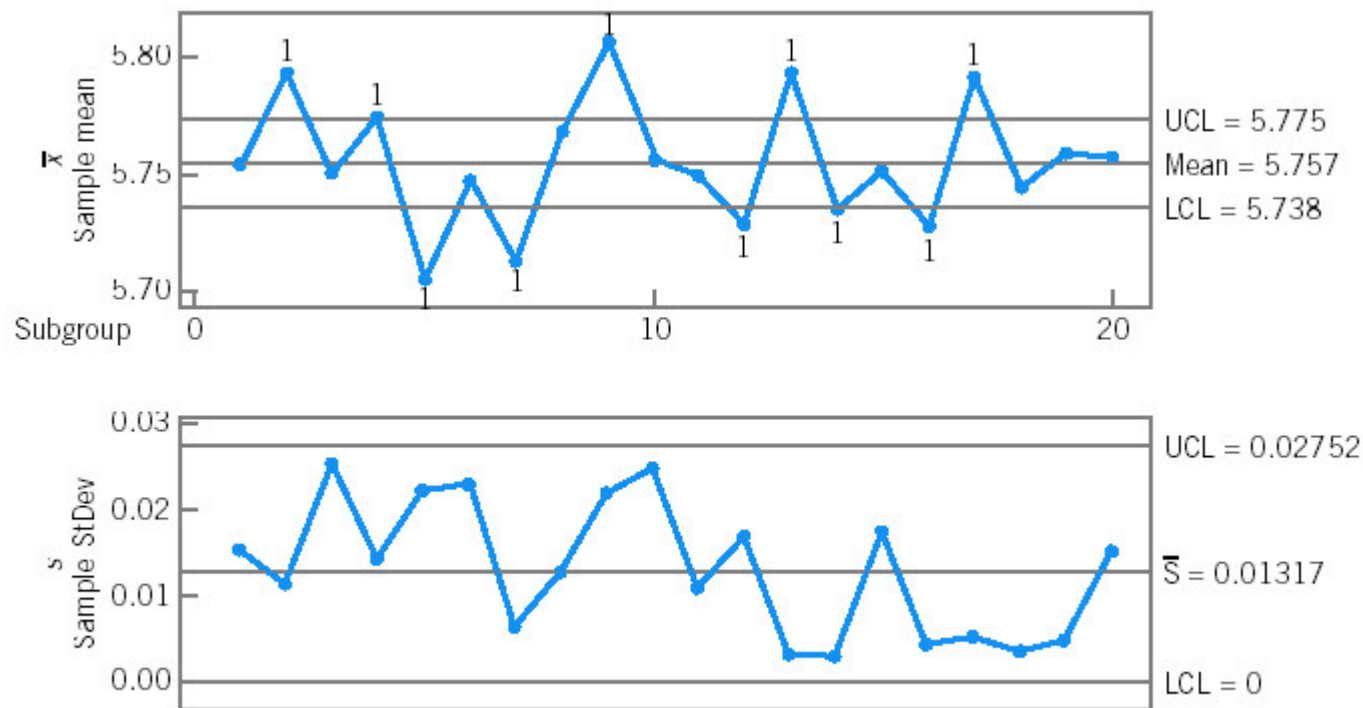


Figure 5-27 Typical $\bar{x}$ and s control charts (from Minitab) for the vane heights of the castings in Fig. 5-26.

Remember that the control chart for $\bar{x}$ deals with the issue of whether or not the between-sample variability is consistent with the within-sample variability. In this case it is not: The vanes on a single casting are formed together in a common wax mold assembly. It is likely that the vane heights on a specific casting will be very similar, and it is reasonable to believe that there will be more variation in average vane height between the castings.

This situation was handled by using the s chart in the ordinary way to measure variation in vane height. However, as this standard deviation is clearly too small to provide a valid basis for control of $\bar{x}$, the quality engineer at the company decided to treat the average vane height on each casting as an *individual measurement* and to control average vane height by using a control chart for individuals with a moving-range chart. This solution worked extremely well in practice, and the group of three control charts provided an excellent basis for process improvement.

Figure 5-28 shows this set of three control charts as generated by Minitab. The Minitab package generates these charts automatically, referring to them as “between/within” control charts. Note that the individuals chart exhibits control, whereas the $\bar{x}$ chart in Fig. 5-27 did not. Essentially the moving range of the average vane heights provides a much more reasonable estimate of the variability in height **between parts**. The s chart can be thought of as a measure of the variability in vane height on a single casting. We want this variability to be as small as possible, so that all vanes on the same part will be nearly identical. The paper by Woodall and Thomas (1995) is a good reference on this general subject.

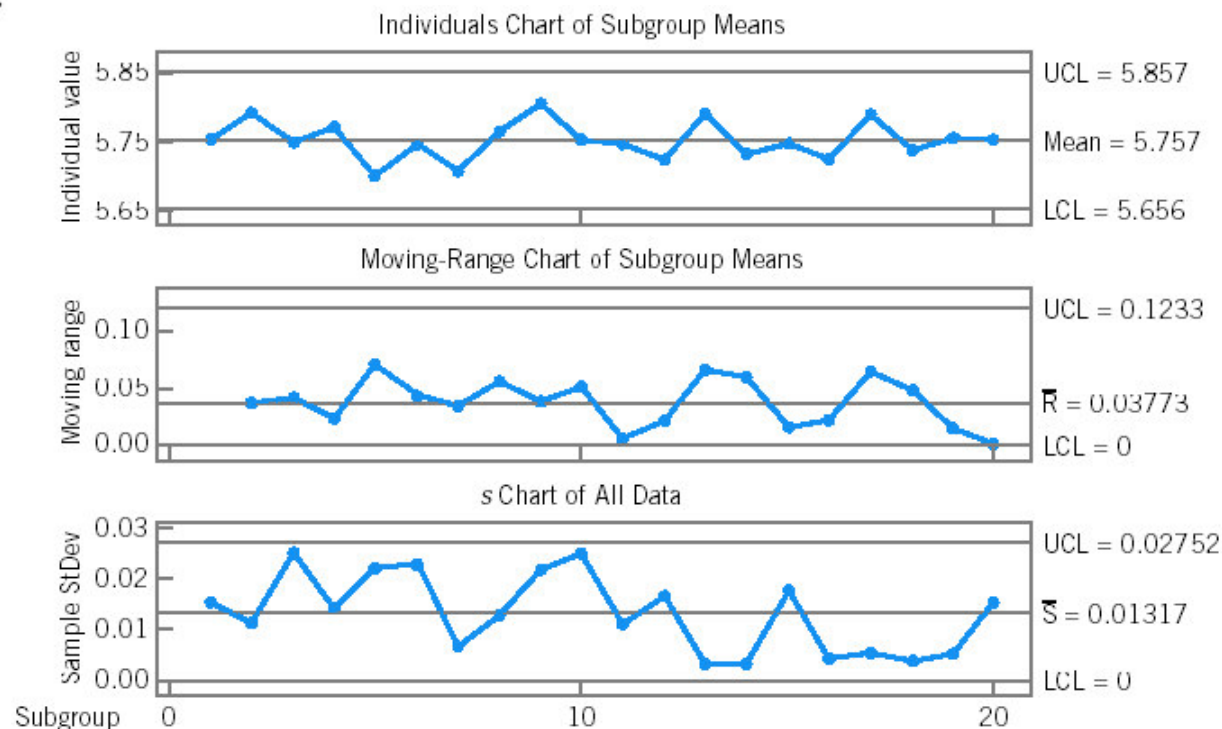


Figure 5-28 Individuals, moving-range, and s control charts for the vane heights of the castings in Fig. 5-26.

IMPORTANT TERMS AND CONCEPTS

Average run length	Process capability ratio (PCR) C_p
Control chart for individuals	R control chart
Interpretation of control charts	Rational subgroups
Moving-range control chart	s control chart
Natural tolerance limits for a process	s^2 control chart
Normality and control charts	Shewhart control charts
Operating characteristic (OC) curve for the $\bar{x}$ control chart	Specification limits
Patterns on control charts	Three-sigma control limits
Phase I control chart usage	Tier chart or tolerance diagram
Phase II control chart usage	Trial control limits
Probability limits for control charts	Variable sample size on control charts
Process capability	Variables control charts
	$\bar{x}$ control chart

LEARNING OBJECTIVES

1. Understand the statistical basis of Shewhart control charts for variables
2. Know how to design variables control charts
3. Know how to set up and use $\bar{x}$ and R control charts
4. Know how to estimate process capability from the control chart information
5. Know how to interpret patterns on $\bar{x}$ and R control charts
6. Know how to set up and use $\bar{x}$ and s or s^2 control charts
7. Know how to set up and use control charts for individual measurements
8. Understand the importance of the normality assumption for individuals control charts and know how to check this assumption
9. Understand the rational subgroup concept for variables control charts
10. Determine the average run length for variables control charts