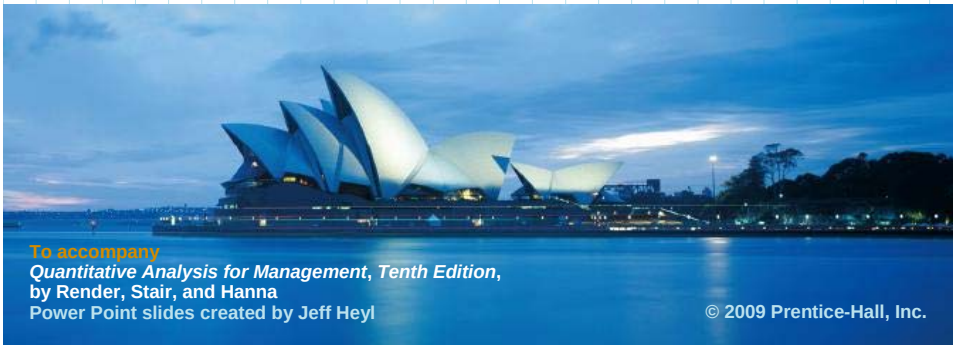


Chapter 9

Linear Programming: The Simplex Method



Learning Objectives

After completing this chapter, students will be able to:

1. Convert LP constraints to equalities with slack, surplus, and artificial variables
2. Set up and solve LP problems with simplex tableaus
3. Interpret the meaning of every number in a simplex tableau
4. Recognize special cases such as infeasibility, unboundedness, and degeneracy
5. Use the simplex tables to conduct sensitivity analysis
6. Construct the dual problem from the primal problem

Chapter Outline

- 9.1 Introduction
- 9.2 How to Set Up the Initial Simplex Solution
- 9.3 Simplex Solution Procedures
- 9.4 The Second Simplex Tableau
- 9.5 Developing the Third Tableau
- 9.6 Review of Procedures for Solving LP Maximization Problems
- 9.7 Surplus and Artificial Variables

Chapter Outline

- 9.8 Solving Minimization Problems
- 9.9 Review of Procedures for Solving LP Minimization Problems
- 9.10 Special Cases
- 9.11 Sensitivity Analysis with the Simplex Tableau
- 9.12 The Dual
- 9.13 Karmarkar's Algorithm

Introduction

- With only two decision variables it is possible to use graphical methods to solve LP problems
- But most real life LP problems are too complex for simple graphical procedures
- We need a more powerful procedure called the *simplex method*
- The simplex method examines the corner points in a systematic fashion using basic algebraic concepts
- It does this in an *iterative* manner until an optimal solution is found
- Each iteration moves us closer to the optimal solution

How To Set Up The Initial Simplex Solution

- Let's look at the Flair Furniture Company from Chapter 7
- This time we'll use the simplex method to solve the problem
- You may recall

T = number of tables produced

C = number of chairs produced

and

$$\begin{array}{ll} \text{Maximize profit} = \$70T + \$50C & \text{(objective function)} \\ \text{subject to} & 2T + 1C \leq 100 \quad \text{(painting hours constraint)} \\ & 4T + 3C \leq 240 \quad \text{(carpentry hours constraint)} \\ & T, C \geq 0 \quad \text{(nonnegativity constraint)} \end{array}$$

Introduction

- Why should we study the simplex method?
- It is important to understand the ideas used to produce solutions
- It provides the optimal solution to the decision variables and the maximum profit (or minimum cost)
- It also provides important economic information
- To be able to use computers successfully and to interpret LP computer printouts, we need to know what the simplex method is doing and why

Converting the Constraints to Equations

- The inequality constraints must be converted into equations
- Less-than-or-equal-to constraints ($\leq$) are converted to equations by adding a *slack variable* to each
- Slack variables represent unused resources
- For the Flair Furniture problem, the slacks are
 - S_1 = slack variable representing unused hours in the painting department
 - S_2 = slack variable representing unused hours in the carpentry department
- The constraints may now be written as

$$2T + 1C + S_1 = 100$$

$$4T + 3C + S_2 = 240$$

Converting the Constraints to Equations

- If the optimal solution uses less than the available amount of a resource, the unused resource is slack
- For example, if Flair produces $T = 40$ tables and $C = 10$ chairs, the painting constraint will be

$$\begin{aligned} 2T + 1C + S_1 &= 100 \\ 2(40) + 1(10) + S_1 &= 100 \\ S_1 &= 10 \end{aligned}$$

- There will be 10 hours of slack, or unused painting capacity

Finding an Initial Solution Algebraically

- There are now two equations and four variables
- When there are more unknowns than equations, you have to set some of the variables equal to 0 and solve for the others
- In this example, two variables must be set to 0 so we can solve for the other two
- A solution found in this manner is called a **basic feasible solution**

Converting the Constraints to Equations

- Each slack variable must appear in every constraint equation
- Slack variables not actually needed for an equation have a coefficient of 0
- So

$$\begin{aligned} 2T + 1C + 1S_1 + 0S_2 &= 100 \\ 4T + 3C + 0S_1 + 1S_2 &= 240 \\ T, C, S_1, S_2 &\geq 0 \end{aligned}$$

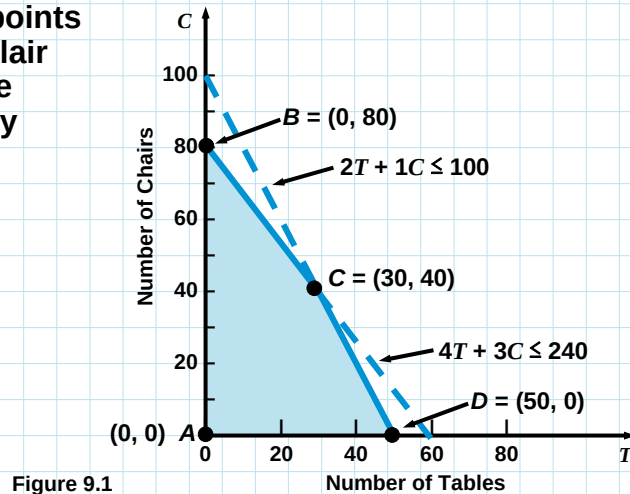
- The objective function becomes
Maximize profit = $\$70T + \$50C + \$0S_1 + \$0S_2$

Finding an Initial Solution Algebraically

- The simplex method starts with an initial feasible solution where all real variables are set to 0
- While this is not an exciting solution, it is a corner point solution
- Starting from this point, the simplex method will move to the corner point that yields the most improved profit
- It repeats the process until it can further improve the solution
- On the following graph, the simplex method starts at point A and then moves to B and finally to C, the optimal solution

Finding an Initial Solution Algebraically

- Corner points for the Flair Furniture Company problem



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The First Simplex Tableau

- The first tableau is called a **simplex tableau**

		Profit per unit column		Production mix column		Real variables		Slack variables		Constant column	
C_j	SOLUTION MIX	\$70 T	\$50 C	\$0 S_1	\$0 S_2	QUANTITY		Profit per unit row			
\$0	S_1	2	1	1	0	100		Constraint equation rows			
\$0	S_2	4	3	0	1	240					
	Z_j	\$0	\$0	\$0	\$0	\$0		Gross profit row			
	$C_j - Z_j$	\$70	\$50	\$0	\$0	\$0		Net profit row			

Table 9.1

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The First Simplex Tableau

- Constraint equations
 - It simplifies handling the LP equations if we put them in tabular form
 - These are the constraint equations for the Flair Furniture problem

SOLUTION MIX	T	C	S_1	S_2	QUANTITY (RIGHT-HAND SIDE)
S_1	2	1	1	0	100
S_2	4	3	0	1	240

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The First Simplex Tableau

- The numbers in the first row represent the coefficients in the first constraint and the numbers in the second the second constraint
- At the initial solution, $T = 0$ and $C = 0$, so $S_1 = 100$ and $S_2 = 240$
- The two slack variables are the **initial solution mix**
- The values are found in the QUANTITY column
- The initial solution is a **basic feasible solution**

$$\begin{bmatrix} T \\ C \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 100 \\ 240 \end{bmatrix}$$

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The First Simplex Tableau

- Variables in the solution mix, called the **basis** in LP terminology, are referred to as **basic variables**
- Variables not in the solution mix or basis (value of 0) are called **nonbasic variables**
- The optimal solution was $T = 30$, $C = 40$, $S_1 = 0$, and $S_2 = 0$
- The final basic variables would be

$$\begin{pmatrix} T \\ C \\ S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 30 \\ 40 \\ 0 \\ 0 \end{pmatrix}$$

The First Simplex Tableau

- Adding the objective function
 - We add a row to the tableau to reflect the objective function values for each variable
 - These contribution rates are called C_j and appear just above each respective variable
 - In the leftmost column, C_j indicates the unit profit for each variable **currently** in the solution mix

C_j		\$70	\$50	\$0	\$0	
C_j	SOLUTION MIX	T	C	S_1	S_2	QUANTITY
\$0	S_1	2	1	1	0	100
\$0	S_2	4	3	0	1	240

The First Simplex Tableau

- Substitution rates
 - The numbers in the body of the tableau are the coefficients of the constraint equations
 - These can also be thought of as **substitution rates**
 - Using the variable T as an example, if Flair were to produce 1 table ($T = 1$), 2 units of S_1 and 4 units of S_2 would have to be removed from the solution
 - Similarly, the substitution rates for C are 1 unit of S_1 and 3 units of S_2
 - Also, for a variable to appear in the solution mix, it must have a 1 someplace in its column and 0s in every other place in that column

The First Simplex Tableau

- The Z_j and $C_j - Z_j$ rows
 - We can complete the initial tableau by adding two final rows
 - These rows provide important economic information including total profit and whether the current solution is optimal
 - We compute the Z_j value by multiplying the contribution value of each number in a column by each number in that row and the j th column, and summing

The First Simplex Tableau

- The Z_j value for the quantity column provides the total contribution of the given solution

$$\begin{aligned} Z_j (\text{gross profit}) &= (\text{Profit per unit of } S_1) \times (\text{Number of units of } S_1) \\ &\quad + (\text{profit per unit of } S_2) \times (\text{Number of units of } S_2) \\ &= \$0 \times 100 \text{ units} + \$0 \times 240 \text{ units} \\ &= \$0 \text{ profit} \end{aligned}$$

- The Z_j values in the other columns represent the gross profit *given up* by adding one unit of this variable into the current solution

$$Z_j = (\text{Profit per unit of } S_1) \times (\text{Substitution rate in row 1}) \\ + (\text{profit per unit of } S_2) \times (\text{Substitution rate in row 2})$$

The First Simplex Tableau

- The $C_j - Z_j$ number in each column represents the net profit that will result from introducing 1 unit of each product or variable into the solution
- It is computed by subtracting the Z_j total for each column from the C_j value at the very top of that variable's column

	COLUMN			
	T	C	S_1	S_2
C_j for column	\$70	\$50	\$0	\$0
Z_j for column	0	0	0	0
$C_j - Z_j$ for column	\$70	\$50	\$0	\$0

The First Simplex Tableau

- Thus,

$$Z_j (\text{for column } T) = (\$0)(2) + (\$0)(4) = \$0$$

$$Z_j (\text{for column } C) = (\$0)(1) + (\$0)(3) = \$0$$

$$Z_j (\text{for column } S_1) = (\$0)(1) + (\$0)(0) = \$0$$

$$Z_j (\text{for column } S_2) = (\$0)(0) + (\$0)(1) = \$0$$

- We can see that no profit is *lost* by adding one unit of either T (tables), C (chairs), S_1 , or S_2

The First Simplex Tableau

- Obviously with a profit of \$0, the initial solution is not optimal
- By examining the numbers in the $C_j - Z_j$ row in Table 9.1, we can see that the total profits can be increased by \$70 for each unit of T and \$50 for each unit of C
- A negative number in the number in the $C_j - Z_j$ row would tell us that the profits would decrease if the corresponding variable were added to the solution mix
- An optimal solution is reached when there are no positive numbers in the $C_j - Z_j$ row

Simplex Solution Procedures

- After an initial tableau has been completed, we proceed through a series of five steps to compute all the numbers needed in the next tableau
- The calculations are not difficult, but they are complex enough that even the smallest arithmetic error can produce a wrong answer

Five Steps of the Simplex Method for Maximization Problems

3. Compute new values for the pivot row. To do this, we simply divide every number in the row by the pivot column.
4. Compute the new values for each remaining row. All remaining rows are calculated as follows:

(New row numbers) = (Numbers in old row)

$$- \left[\begin{array}{c} \text{Number above} \\ \text{or below} \\ \text{pivot number} \end{array} \right] \times \left[\begin{array}{c} \text{Corresponding number in} \\ \text{the new row, that is, the} \\ \text{row replaced in step 3} \end{array} \right]$$

Five Steps of the Simplex Method for Maximization Problems

1. Determine the variable to enter the solution mix next. One way of doing this is by identifying the column, and hence the variable, with the largest positive number in the $C_j - Z_j$ row of the preceding tableau. The column identified in this step is called the **pivot column**.
2. Determine which variable to replace. This is accomplished by dividing the quantity column by the corresponding number in the column selected in step 1. The row with the smallest nonnegative number calculated in this fashion will be replaced in the next tableau. This row is often referred to as the **pivot row**. The number at the intersection of the pivot row and pivot column is the **pivot number**.

Five Steps of the Simplex Method for Maximization Problems

5. Compute the Z_j and $C_j - Z_j$ rows, as demonstrated in the initial tableau. If all the numbers in the $C_j - Z_j$ row are 0 or negative, an optimal solution has been reached. If this is not the case, return to step 1.

The Second Simplex Tableau

- We can now apply these steps to the Flair Furniture problem

Step 1. Select the variable with the largest positive $C_j - Z_j$ value to enter the solution next. In this case, variable T with a contribution value of \$70.

C_j			\$70	\$50	\$0	\$0	
	SOLUTION MIX		T	C	S_1	S_2	QUANTITY (RHS)
\$0	S_1	2	1	1	0	100	
\$0	S_2	4	3	0	1	240	
	Z_j	\$0	\$0	\$0	\$0	\$0	\$0
	$C_j - Z_j$	\$70	\$50	\$0	\$0		total profit

Pivot column

Table 9.2

The Second Simplex Tableau

We choose the smaller ratio (50) and this determines the S_1 variable is to be replaced. This corresponds to point D on the graph in Figure 9.2.

C_j			\$70	\$50	\$0	\$0	
	SOLUTION MIX		T	C	S_1	S_2	QUANTITY (RHS)
\$0	S_1	2	1	1	0	100	
\$0	S_2	4	3	0	1	240	
	Z_j	\$0	\$0	\$0	\$0	\$0	\$0
	$C_j - Z_j$	\$70	\$50	\$0	\$0		

Pivot number

Pivot row

Pivot column

Table 9.3

The Second Simplex Tableau

Step 2. Select the variable to be replaced. Either S_1 or S_2 will have to leave to make room for T in the basis. The following ratios need to be calculated.

For the S_1 row

$$\frac{100(\text{hours of painting time available})}{2(\text{hours required per table})} = 50 \text{ tables}$$

For the S_2 row

$$\frac{240(\text{hours of carpentry time available})}{4(\text{hours required per table})} = 60 \text{ tables}$$

The Second Simplex Tableau

Step 3. We can now begin to develop the second, improved simplex tableau. We have to compute a replacement for the pivot row. This is done by dividing every number in the pivot row by the pivot number. The new version of the pivot row is below.

$$\frac{2}{2} = 1 \quad \frac{1}{2} = 0.5 \quad \frac{1^*}{2} = 0.5 \quad \frac{0}{2} = 0 \quad \frac{100}{2} = 50$$

C_j	SOLUTION MIX	T	C	S_1	S_2	QUANTITY
\$70	T	1	0.5	0.5	0	50

The Second Simplex Tableau

Step 4. Completing the rest of the tableau, the S_2 row, is slightly more complicated. The right of the following expression is used to find the left side.

$$\left[\begin{array}{c} \text{Number in} \\ \text{New } S_2 \text{ Row} \end{array} \right] = \left[\begin{array}{c} \text{Number in} \\ \text{Old } S_2 \text{ Row} \end{array} \right] - \left[\left[\begin{array}{c} \text{Number Below} \\ \text{Pivot Number} \end{array} \right] \times \left[\begin{array}{c} \text{Corresponding Number} \\ \text{in the New } T \text{ Row} \end{array} \right] \right]$$

0	=	4	-	(4)	×	(1)
1	=	3	-	(4)	×	(0.5)
-2	=	0	-	(4)	×	(0.5)
1	=	1	-	(4)	×	(0)
40	=	240	-	(4)	×	(50)

C_j	SOLUTION MIX	T	C	S_1	S_2	QUANTITY
\$70	T	1	0.5	0.5	0	50
\$0	S_2	0	1	-2	1	40

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The Second Simplex Tableau

Step 5. The final step of the second iteration is to introduce the effect of the objective function. This involves computing the $C_j - Z_j$ rows. The Z_j for the quantity row gives us the gross profit and the other Z_j represent the gross profit given up by adding one unit of each variable into the solution.

$$Z_j \text{ (for } T \text{ column)} = (\$70)(1) + (\$0)(0) = \$70$$

$$Z_j \text{ (for } C \text{ column)} = (\$70)(0.5) + (\$0)(1) = \$35$$

$$Z_j \text{ (for } S_1 \text{ column)} = (\$70)(0.5) + (\$0)(-2) = \$35$$

$$Z_j \text{ (for } S_2 \text{ column)} = (\$70)(0) + (\$0)(1) = \$0$$

$$Z_j \text{ (for total profit)} = (\$70)(50) + (\$0)(40) = \$3,500$$

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The Second Simplex Tableau

The T column contains $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the S_2 column contains $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, necessary conditions for variables to be in the solution. The manipulations of steps 3 and 4 were designed to produce 0s and 1s in the appropriate positions.

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The Second Simplex Tableau

	COLUMN			
	T	C	S_1	S_2
C_j for column	\$70	\$50	\$0	\$0
Z_j for column	\$70	\$35	\$35	\$0
$C_j - Z_j$ for column	\$0	\$15	-\$35	\$0

Completed second simplex tableau

C_j		\$70	\$50	\$0	\$0	
	SOLUTION MIX	T	C	S_1	S_2	QUANTITY (RHS)
\$0	T	1	0.5	0.5	0	50
\$0	S_2	0	1	-2	1	40
	Z_j	\$70	\$35	\$35	\$0	\$3,500
	$C_j - Z_j$	\$0	\$15	-\$35	\$0	

Table 9.4

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Interpreting the Second Tableau

■ Current solution

- The solution point of 50 tables and 0 chairs ($T = 50, C = 0$) generates a profit of \$3,500. T is a basic variable and C is a nonbasic variable. This corresponds to point D in Figure 9.2.

■ Resource information

- Slack variable S_2 is the unused time in the carpentry department and is in the basis. Its value implies there is 40 hours of unused carpentry time remaining. Slack variable S_1 is nonbasic and has a value of 0 meaning there is no slack time in the painting department.

Interpreting the Second Tableau

■ Net profit row

- The $C_j - Z_j$ row is important for two reasons
- First, it indicates whether the current solution is optimal
- When there are no positive values in the bottom row, an optimal solution to a maximization LP has been reached
- The second reason is that we use this row to determine which variable will enter the solution next

Interpreting the Second Tableau

■ Substitution rates

- Substitution rates are the coefficients in the heart of the tableau. In column C , if 1 unit of C is added to the current solution, 0.5 units of T and 1 unit of S_2 must be given up. This is because the solution $T = 50$ uses up all 100 hours of painting time available.
- Because these are **marginal** rates of substitution, so only 1 more unit of S_2 is needed to produce 1 chair
- In column S_1 , the substitution rates mean that if 1 hour of slack painting time is added to producing a chair, 0.5 **less** of a table will be produced

Developing the Third Tableau

- Since the previous tableau is not optimal, we repeat the five simplex steps

Step 1. Variable C will enter the solution as its $C_j - Z_j$ value of 15 is the largest positive value. The C column is the new pivot column.

Step 2. Identify the pivot row by dividing the number in the quantity column by its corresponding substitution rate in the C column.

$$\text{For the } T \text{ row : } \frac{50}{0.5} = 100 \text{ chairs}$$

$$\text{For the } S_2 \text{ row : } \frac{40}{1} = 40 \text{ chairs}$$

Developing the Third Tableau

These ratios correspond to the values of C at points F and C in Figure 9.2. The S_2 row has the smallest ratio so S_2 will leave the basis and will be replaced by C .

C_j	SOLUTION MIX	T	C	S_1	S_2	QUANTITY
\$70	T	1	0.5	0.5	0	50
\$0	S_2	0	①	-2	1	40
		Pivot number				Pivot row
Z_j	\$70	\$35	\$35	\$0	\$3,500	
$C_j - Z_j$	\$0	\$15	-\$35	\$0		
		Pivot column				

Table 9.5

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Developing the Third Tableau

Step 4. The new values for the T row may now be computed

$$\left[\begin{array}{c} \text{Number in} \\ \text{new } T \text{ row} \end{array} \right] = \left[\begin{array}{c} \text{Number in} \\ \text{old } T \text{ row} \end{array} \right] - \left[\left[\begin{array}{c} \text{Number above} \\ \text{pivot number} \end{array} \right] \times \left[\begin{array}{c} \text{Corresponding number} \\ \text{in new } C \text{ row} \end{array} \right] \right]$$

1	=	1	-	(0.5)	×	(0)
0	=	0.5	-	(0.5)	×	(1)
1.5	=	0.5	-	(0.5)	×	(-2)
-0.5	=	0	-	(0.5)	×	(1)
30	=	50	-	(0.5)	×	(40)

C_j	SOLUTION MIX	T	C	S_1	S_2	QUANTITY
\$70	T	1	0	1.5	-0.5	30
\$50	C	0	1	-2	1	40

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Developing the Third Tableau

Step 3. The pivot row is replaced by dividing every number in it by the pivot point number

$$\frac{0}{1} = 0 \quad \frac{1}{1} = 1 \quad \frac{-2}{1} = -2 \quad \frac{1}{1} = 1 \quad \frac{40}{1} = 40$$

The new C row is

C_j	SOLUTION MIX	T	C	S_1	S_2	QUANTITY
\$5	C	0	1	-2	1	40

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Developing the Third Tableau

Step 5. The Z_j and $C_j - Z_j$ rows can now be calculated

$$\begin{aligned} Z_j \text{ (for } T \text{ column)} &= (\$70)(1) + (\$50)(0) = \$70 \\ Z_j \text{ (for } C \text{ column)} &= (\$70)(0) + (\$50)(1) = \$50 \\ Z_j \text{ (for } S_1 \text{ column)} &= (\$70)(1.5) + (\$50)(-2) = \$5 \\ Z_j \text{ (for } S_2 \text{ column)} &= (\$70)(-0.5) + (\$50)(1) = \$15 \\ Z_j \text{ (for total profit)} &= (\$70)(30) + (\$50)(40) = \$4,100 \end{aligned}$$

And the net profit per unit row is now

	COLUMN			
	T	C	S_1	S_2
C_j for column	\$70	\$50	\$0	\$0
Z_j for column	\$70	\$50	\$5	\$15
$C_j - Z_j$ for column	\$0	\$0	-\$5	-\$15

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Developing the Third Tableau

- Note that every number in the $C_j - Z_j$ row is 0 or negative indicating an optimal solution has been reached

- The optimal solution is

$T = 30$ tables

$C = 40$ chairs

$S_1 = 0$ slack hours in the painting department

$S_2 = 0$ slack hours in the carpentry department

profit = \$4,100 for the optimal solution

Review of Procedures for Solving LP Maximization Problems

- I. Formulate the LP problem's objective function and constraints
- II. Add slack variables to each less-than-or-equal-to constraint and to the objective function
- III. Develop an initial simplex tableau with slack variables in the basis and decision variables set equal to 0. compute the Z_j and $C_j - Z_j$ values for this tableau.
- IV. Follow the five steps until an optimal solution has been reached

Developing the Third Tableau

- The final simplex tableau for the Flair Furniture problem corresponds to point C in Figure 9.2

C_j ↓	SOLUTION MIX	C_j →				QUANTITY
		\$70	\$50	\$0	\$0	
	T	1	0	1.5	-0.5	30
	C	0	1	-2	1	40
	Z_j	\$70	\$50	\$5	\$15	\$4,100
	$C_j - Z_j$	\$0	\$0	-\$5	-\$15	

Table 9.6

- Arithmetic mistakes are easy to make
- It is always a good idea to check your answer by going back to the original constraints and objective function

Review of Procedures for Solving LP Maximization Problems

1. Choose the variable with the greatest positive $C_j - Z_j$ to enter the solution in the pivot column.
2. Determine the solution mix variable to be replaced and the pivot row by selecting the row with the smallest (nonnegative) ratio of the quantity-to-pivot column substitution rate.
3. Calculate the new values for the pivot row
4. Calculate the new values for the other row(s)
5. Calculate the Z_j and $C_j - Z_j$ values for this tableau. If there are any $C_j - Z_j$ numbers greater than 0, return to step 1. If not, an optimal solution has been reached.

Surplus and Artificial Variables

- Greater-than-or-equal-to ($\geq$) constraints are just as common in real problems as less-than-or-equal-to ($\leq$) constraints and equalities
- To use the simplex method with these constraints, they must be converted to a special form similar to that made for the less-than-or-equal-to ($\leq$) constraints
- If they are not, the simplex technique is unable to set up an initial solution in the first tableau
- Consider the following two constraints

$$\text{Constraint 1: } 5X_1 + 10X_2 + 8X_3 \geq 210$$

$$\text{Constraint 2: } 25X_1 + 30X_2 = 900$$

Surplus and Artificial Variables

- To convert the first constraint we subtract a surplus variable, S_1 , to create an equality
- Constraint 1 rewritten : $5X_1 + 10X_2 + 8X_3 - S_1 = 210$
- If we solved this for $X_1 = 20$, $X_2 = 8$, $X_3 = 5$, S_1 would be

$$5X_1 + 10X_2 + 8X_3 - S_1 = 210$$

$$5(20) + 10(8) + 8(5) - S_1 = 210$$

$$100 + 80 + 40 - S_1 = 210$$

$$-S_1 = 210 - 220$$

$$S_1 = 10 \text{ surplus units}$$

Surplus and Artificial Variables

- Surplus variables
 - Greater-than-or-equal-to ($\geq$) constraints require a different approach than the less-than-or-equal-to ($\leq$) constraints we have seen
 - They involve the subtraction of a **surplus variable** rather than the addition of a slack variable
 - The surplus variable tells us how much the solution exceeds the constraint amount
 - This is sometimes called **negative slack**

Surplus and Artificial Variables

- Artificial variables
 - There is one more step in this process
 - If a surplus variable is added by itself, it would have a negative value in the initial tableau where all real variables are set to zero

$$5(0) + 10(0) + 8(0) - S_1 = 210$$

$$0 - S_1 = 210$$

$$S_1 = -210$$

- But **all** variables in LP problems **must** be nonnegative at all times

Surplus and Artificial Variables

- To resolve this we add in another variable called an **artificial variable**

Constraint 1 completed : $5X_1 + 10X_2 + 8X_3 - S_1 + A_1 = 210$

- Now X_1 , X_2 , X_3 , and S_1 can all be 0 in the initial solution and A_1 will equal 210
- The same situation applies in equality constraint equations as well

Constraint 2 rewritten : $25X_1 + 30X_2 + A_2 = 900$

Surplus and Artificial Variables

- Surplus and artificial variables in the objective function

- Both types of variables must be included in the objective function
- Surplus variables, like slack variables, carry a \$0 cost coefficient
- Since artificial variables must be forced out of the solution, we assign an arbitrarily high cost
- By convention we use the coefficient M (or $-M$ in maximization problems) which simply represents a very large number

Surplus and Artificial Variables

- Artificial variables are inserted into equality constraints so we can easily develop an initial feasible solution
- When a problem has many constraint equations with many variables, it is not possible to “eyeball” an initial solution
- Using artificial variables allows us to use the automatic initial solution of setting all the other variables to 0
- Unlike slack or surplus variables, artificial variables have no meaning in the problem formulation
- They are strictly a computational tool, they will be gone in the final solution

Surplus and Artificial Variables

- A problem with this objective function

Minimize cost = $\$5X_1 + \$9X_2 + \$7X_3$

And the constraint equations we saw before would appear as follows:

Minimize cost = $\$5X_1 + \$9X_2 + \$7X_3 + \$0S_1 + \$MA_1 + \MA_2
 subject to $5X_1 + 10X_2 + 8X_3 - 1S_1 + 1A_1 + 0A_2 = 210$
 $25X_1 + 30X_2 + 0X_3 + 0S_1 + 0A_1 + 1A_2 = 900$

Solving Minimization Problems

- Once the necessary equations are developed for a minimization problem, we can use the simplex method to solve for an optimal solution

The Muddy River Chemical Corporation Example

- The model formulation would be

$$\begin{aligned} \text{Minimize cost} &= \$5X_1 + \$6X_2 \\ \text{subject to} \quad &X_1 + X_2 = 1,000 \text{ lb} \\ &X_1 \leq 300 \text{ lb} \\ &X_2 \geq 150 \text{ lb} \\ &X_1, X_2 \geq 0 \end{aligned}$$

where

X_1 = number of pounds of phosphate

X_2 = number of pounds of potassium

The Muddy River Chemical Corporation Example

- The Muddy River Chemical Corporation must produce exactly 1,000 pounds of a special mixture of phosphate and potassium for a customer
- Phosphate costs \$5 per pound and potassium \$6 per pound
- No more than 300 pounds of phosphate can be used and at least 150 pounds of potassium must be used
- The company wants to find the least-cost blend of the two ingredients

The Muddy River Chemical Corporation Example

- Graphical analysis**
 - Because there are only two decision variables, we can plot the constraints and the feasible region as shown in Figure 9.3
 - Because $X_1 + X_2 = 1,000$ is an equality, the optimal solution must lie on this line
 - It must also lie between points A and B because of the $X_1 \leq 300$ constraint
 - It turns out the $X_2 \geq 150$ is redundant and nonbinding
 - The optimal corner point is point B (300, 700) for a total cost of \$5,700

The Muddy River Chemical Corporation Example

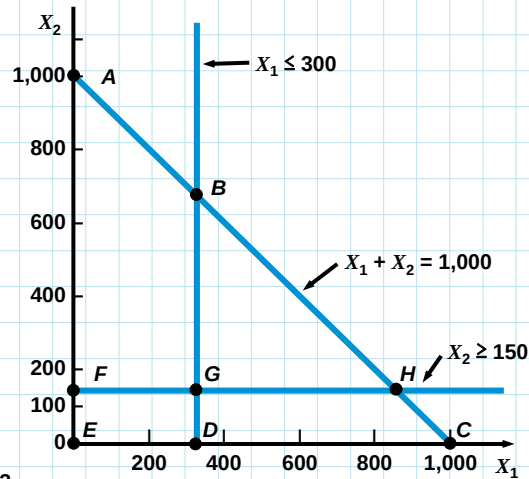


Figure 9.3

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The Muddy River Chemical Corporation Example

■ Converting the constraints and objective function

- The necessary artificial variables, slack variables, and surplus variables need to be added to the equations
- The revised model is

$$\begin{aligned}
 \text{Minimize cost} &= \$5X_1 + \$6X_2 + \$0S_1 + \$0S_2 + \$MA_1 + \$MA_2 \\
 \text{subject to} \quad &1X_1 + 1X_2 + 0S_1 + 0S_2 + 1A_1 + 0A_2 = 1,000 \\
 &1X_1 + 0X_2 + 1S_1 + 0S_2 + 0A_1 + 0A_2 = 300 \\
 &0X_1 + 1X_2 + 0S_1 - 1S_2 + 0A_1 + 1A_2 = 150 \\
 &X_1, X_2, S_1, S_2, A_1, A_2 \geq 0
 \end{aligned}$$

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The Muddy River Chemical Corporation Example

- Rarely will problems be this simple
- The simplex method can be used to solve much more complex problems
- In this example, the simplex method will start at corner point E, move to point F, then G and finally to point B which is the optimal solution

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Rules of the Simplex Method for Minimization Problems

- Minimization problems are quite similar to the maximization problems tackled earlier
- The significant difference is the $C_j - Z_j$ row
- We will now choose the variable with the **negative** $C_j - Z_j$ that gives the largest improvement
- We select the variable that decreases costs the most
- In minimization problems, an optimal solution is reached when all the numbers in the $C_j - Z_j$ are 0 or **positive**
- All other steps in the simplex method remain the same

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Steps for Simplex Minimization Problems

1. Choose the variable with the greatest negative $C_j - Z_j$ to enter the solution in the pivot column.
2. Determine the solution mix variable to be replaced and the pivot row by selecting the row with the smallest (nonnegative) ratio of the quantity-to-pivot column substitution rate.
3. Calculate the new values for the pivot row
4. Calculate the new values for the other row(s)
5. Calculate the Z_j and $C_j - Z_j$ values for this tableau. If there are any $C_j - Z_j$ numbers less than 0, return to step 1. if not, and optimal solution has been reached.

First Simplex Tableau for the Muddy River Chemical Corporation Example

- The numbers in the Z_j are computed by multiplying the C_j column on the far left of the table times the corresponding numbers in each other column

$$\begin{aligned}
 Z_j \text{ (for } X_1 \text{ column)} &= \$M(1) + \$0(1) + \$M(0) = \$M \\
 Z_j \text{ (for } X_2 \text{ column)} &= \$M(1) + \$0(0) + \$M(1) = \$2M \\
 Z_j \text{ (for } S_1 \text{ column)} &= \$M(0) + \$0(1) + \$M(0) = \$0 \\
 Z_j \text{ (for } S_2 \text{ column)} &= \$M(0) + \$0(0) + \$M(-1) = -\$M \\
 Z_j \text{ (for } A_1 \text{ column)} &= \$M(1) + \$0(0) + \$M(0) = \$M \\
 Z_j \text{ (for } A_2 \text{ column)} &= \$M(0) + \$0(0) + \$M(1) = \$M \\
 Z_j \text{ (for total cost)} &= \$M(1,000) + \$0(300) + \$M(150) = \$1,150M
 \end{aligned}$$

First Simplex Tableau for the Muddy River Chemical Corporation Example

- The initial tableau is set up in the same manner as the in the maximization problem
- The first three rows are
- Note the costs for the artificial variables are $\$M$
- We simply treat this as a very large number which forces the artificial variables out of the solution quickly

C_j	SOLUTION MIX	X_1	X_2	S_1	S_2	A_1	A_2	QUANTITY
$\$M$	A_1	1	1	0	0	1	0	1,000
$\$0$	S_1	1	0	1	0	0	0	300
$\$M$	A_2	0	1	0	-1	0	1	150

First Simplex Tableau for the Muddy River Chemical Corporation Example

- The $C_j - Z_j$ entires are determined as follows

	COLUMN					
	X_1	X_2	S_1	S_2	A_1	A_2
C_j for column	$\$5$	$\$6$	$\$0$	$\$0$	$\$M$	$\$M$
Z_j for column	$\$M$	$\$2M$	$\$0$	$-\$M$	$\$M$	$\$M$
$C_j - Z_j$ for column	$-\$M + \5	$-\$2M + \6	$\$0$	$\$M$	$\$0$	$\$0$

First Simplex Tableau for the Muddy River Chemical Corporation Example

- The initial solution was obtained by letting each of the variables X_1 , X_2 , and S_2 assume a value of 0
- The current basic variables are $A_1 = 1,000$, $S_1 = 150$, and $A_2 = 150$
- The complete solution could be expressed in vector form as

$$\begin{pmatrix} X_1 \\ X_2 \\ S_1 \\ S_2 \\ A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 300 \\ 0 \\ 1,000 \\ 150 \end{pmatrix}$$

Developing the Second Tableau

- In the $C_j - Z_j$ row there are two entries with negative values, X_1 and X_2
- This means an optimal solution does not yet exist
- The negative entry for X_2 indicates it has the will result in the largest improvement, which means it will enter the solution next
- To find the variable that will leave the solution, we divide the elements in the quantity column by the respective pivot column substitution rates

First Simplex Tableau for the Muddy River Chemical Corporation Example

- The initial tableau

C_j →		\$5	\$6	\$0	\$0	\$M	\$M	
	SOLUTION MIX	X_1	X_2	S_1	S_2	A_1	A_2	QUANTITY
\$M	A_1	1	1	0	0	1	0	1,000
\$0	S_1	1	0	1	0	0	0	300
\$M	A_2	0	①	0	-1	0	1	150
				Pivot number				Pivot row
	Z_j	\$M	\$M	\$0	-\$M	\$M	\$M	\$1,150M
	$C_j - Z_j$	-\$M + \$5	-2M + \$6	\$0	\$M	\$0	\$0	
				Pivot column				

Table 9.7

Developing the Second Tableau

$$\text{For the } A_1 \text{ row} = \frac{1,000}{1} = 1,000$$

$$\text{For the } S_1 \text{ row} = \frac{300}{0} \quad (\text{this is an undefined ratio, so we ignore it})$$

$$\text{For the } A_2 \text{ row} = \frac{150}{1} = 150 \quad (\text{smallest quotient, indicating pivot row})$$

- Hence the pivot row is the A_2 row and the pivot number is at the intersection of the X_2 column and the A_2 row

Developing the Second Tableau

- The entering row for the next tableau is found by dividing each element in the pivot row by the pivot number

(New row numbers) = (Numbers in old row)

$$- \left[\begin{array}{c} \text{Number above or} \\ \text{below pivot number} \end{array} \right] \times \left[\begin{array}{c} \text{Corresponding number} \\ \text{in newly replaced row} \end{array} \right]$$

A_1 Row	S_1 Row
$1 = 1 - (1)(0)$	$1 = 1 - (0)(0)$
$0 = 1 - (1)(1)$	$0 = 0 - (0)(1)$
$0 = 0 - (1)(0)$	$1 = 1 - (0)(0)$
$1 = 0 - (1)(-1)$	$0 = 0 - (0)(-1)$
$1 = 1 - (1)(0)$	$0 = 0 - (0)(0)$
$-1 = 0 - (1)(1)$	$0 = 0 - (0)(1)$
$850 = 1,000 - (1)(150)$	$300 = 300 - (0)(150)$

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Developing the Second Tableau

- Second simplex tableau

C_j		\$5	\$6	\$0	\$0	\$M	\$M	
	SOLUTION MIX	X_1	X_2	S_1	S_2	A_1	A_2	QUANTITY
\$M	A_1	1	0	0	1	1	-1	850
\$0	S_1	1	0	1	0	0	0	300
		Pivot number						Pivot row
\$6	X_2	0	1	0	-1	0	1	150
	Z_j	\$M	\$6	\$0	$\$M - 6$	\$M	$-\$M + 6$	$\$850M + \900
	$C_j - Z_j$	$-\$M + \5	\$0	\$0	$-\$M + \6	\$0	$\$2M - 6$	
		Pivot column						

Table 9.8

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Developing the Second Tableau

- The Z_j and $C_j - Z_j$ rows are computed next

Z_j (for X_1)	=	$\$M(1)$	+ $\$0(1)$	+ $\$6(0)$	=	$\$M$
Z_j (for X_2)	=	$\$M(0)$	+ $\$0(0)$	+ $\$6(1)$	=	$\$6$
Z_j (for S_1)	=	$\$M(0)$	+ $\$0(1)$	+ $\$6(0)$	=	$\$0$
Z_j (for S_2)	=	$\$M(1)$	+ $\$0(0)$	+ $\$6(-1)$	=	$\$M - 6$
Z_j (for A_1)	=	$\$M(1)$	+ $\$0(0)$	+ $\$6(0)$	=	$\$M$
Z_j (for A_2)	=	$\$M(-1)$	+ $\$0(0)$	+ $\$6(1)$	=	$-\$M + 6$
Z_j (for total cost)	=	$\$M(850) + \$0(300) + \$6(150) = \$850M + \$900$				

	COLUMN					
	X_1	X_2	S_1	S_2	A_1	A_2
C_j for column	\$5	\$6	\$0	\$0	\$M	\$M
Z_j for column	\$M	\$6	\$0	$\$M - 6$	\$M	$-\$M + 6$
$C_j - Z_j$ for column	$-\$M + \5	\$0	\$0	$-\$M + 6$	\$0	$\$2M - 6$

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Developing a Third Tableau

- The new pivot column is the X_1 column and we check the **quantity column-to-pivot column** ratio

$$\text{For the } A_1 \text{ row} = \frac{850}{1} = 850$$

$$\text{For the } S_1 \text{ row} = \frac{300}{1} = 300 \quad (\text{smallest ratio})$$

$$\text{For the } X_2 \text{ row} = \frac{150}{0} = \text{undefined}$$

- Hence variable S_1 will be replaced by X_1

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Developing a Third Tableau

- To replace the pivot row we divide each number in the S_1 row by 1 leaving it unchanged
- The other calculations are shown below

A_1 Row	S_1 Row
$0 = 1 - (1)(1)$	$0 = 0 - (0)(1)$
$0 = 0 - (1)(0)$	$1 = 1 - (0)(0)$
$-1 = 0 - (1)(1)$	$0 = 0 - (0)(1)$
$1 = 1 - (1)(0)$	$-1 = -1 - (0)(0)$
$1 = 1 - (1)(0)$	$0 = 0 - (0)(0)$
$-1 = -1 - (1)(0)$	$1 = 1 - (0)(0)$
$550 = 850 - (1)(300)$	$150 = 150 - (0)(300)$

Developing a Third Tableau

- The third simplex tableau for the Muddy River Chemical problem

C_j		\$5	\$6	\$0	\$0	\$M	\$M	
	SOLUTION MIX	X_1	X_2	S_1	S_2	A_1	A_2	QUANTITY
\$M	A_1	0	0	-1	1	1	-1	550
\$5	X_1	1	0	1	0	0	0	300
\$6	X_2	0	1	0	-1	0	1	150
	Z_j	\$5	\$6	-\$M + 5	\$M - 6	\$M	-\$M + 6	\$550M + 2,400
	$C_j - Z_j$	\$0	\$0	\$M - 5	-\$M + 6	\$0	\$2M - 6	

Pivot number: 1 (in S_2 column, A_1 row)
Pivot row: A_1 row
Pivot column: S_2 column

Table 9.9

Developing a Third Tableau

- The Z_j and $C_j - Z_j$ rows are computed next

Z_j (for X_1)	=	$\$M(0)$	+ $\$5(1)$	+ $\$6(0)$	=	$\$5$
Z_j (for X_2)	=	$\$M(0)$	+ $\$5(0)$	+ $\$6(1)$	=	$\$6$
Z_j (for S_1)	=	$\$M(-1)$	+ $\$5(1)$	+ $\$6(0)$	=	$-\$M + 5$
Z_j (for S_2)	=	$\$M(1)$	+ $\$5(0)$	+ $\$6(-1)$	=	$\$M - 6$
Z_j (for A_1)	=	$\$M(1)$	+ $\$5(0)$	+ $\$6(0)$	=	$\$M$
Z_j (for A_2)	=	$\$M(-1)$	+ $\$5(0)$	+ $\$6(1)$	=	$-\$M + 6$
Z_j (for total cost)	=	$\$M(550)$	+ $\$5(300)$	+ $\$6(150)$	=	$\$550M + 2,400$

	COLUMN					
	X_1	X_2	S_1	S_2	A_1	A_2
C_j for column	\$5	\$6	\$0	\$0	\$M	\$M
Z_j for column	\$5	\$6	-\$M + 5	\$M - 6	\$M	-\$M + 6
$C_j - Z_j$ for column	\$0	\$0	\$M + 5	-\$M + 6	\$0	\$2M - 6

Fourth Tableau for Muddy River

- The new pivot column is the S_2 column

For the A_1 row = $\frac{550}{1} = 550$ (row to be replaced)

For the X_1 row = $\frac{300}{0}$ (undefined)

For the X_2 row = $\frac{150}{-1}$ (not considered because it is negative)

Fourth Tableau for Muddy River

- Each number in the pivot row is again divided by 1
- The other calculations are shown below

X_1 Row	X_2 Row
$1 = 1 - (0)(0)$	$0 = 0 - (-1)(0)$
$0 = 0 - (0)(0)$	$1 = 1 - (-1)(0)$
$1 = 1 - (0)(-1)$	$-1 = 0 - (-1)(-1)$
$0 = 0 - (0)(1)$	$0 = -1 - (-1)(1)$
$0 = 0 - (0)(1)$	$1 = 0 - (-1)(1)$
$0 = 0 - (0)(-1)$	$0 = 1 - (-1)(-1)$
$300 = 300 - (0)(550)$	$700 = 150 - (-1)(550)$

Fourth Tableau for Muddy River

- Fourth and optimal tableau for the Muddy River Chemical Corporation problem

C_j		\$5	\$6	\$0	\$0	\$M	\$M	
	SOLUTION MIX	X_1	X_2	S_1	S_2	A_1	A_2	QUANTITY
\$0	S_2	0	0	-1	1	1	-1	550
\$5	X_1	1	0	1	0	0	0	300
\$6	X_2	0	1	-1	0	1	0	700
	Z_j	\$5	\$6	-\$1	\$0	\$6	\$0	\$5,700
	$C_j - Z_j$	\$0	\$0	\$1	\$0	$M - 6$	M	

Table 9.10

Fourth Tableau for Muddy River

- Finally the Z_j and $C_j - Z_j$ rows are computed

Z_j (for X_1)	=	\$0(0)	+ \$5(1)	+ \$6(0)	=	\$5
Z_j (for X_2)	=	\$0(0)	+ \$5(0)	+ \$6(1)	=	\$6
Z_j (for S_1)	=	\$0(-1)	+ \$5(1)	+ \$6(-1)	=	-\$1
Z_j (for S_2)	=	\$0(1)	+ \$5(0)	+ \$6(0)	=	\$0
Z_j (for A_1)	=	\$0(1)	+ \$5(0)	+ \$6(1)	=	\$6
Z_j (for A_2)	=	\$0(-1)	+ \$5(0)	+ \$6(0)	=	\$0
Z_j (for total cost)	=	\$0(550)	+ \$5(300)	+ \$6(700)	=	\$5,700

	COLUMN					
	X_1	X_2	S_1	S_2	A_1	A_2
C_j for column	\$5	\$6	\$0	\$0	M	M
Z_j for column	\$5	\$6	-\$1	\$0	\$6	\$0
$C_j - Z_j$ for column	\$0	\$0	\$1	\$0	$M - 6$	M

Review of Procedures for Solving LP Minimization Problems

- Formulate the LP problem's objective function and constraints
- Include slack variables to each less-than-or-equal-to constraint and both surplus and artificial variables to greater-than-or-equal-to constraints and add all variables to the objective function
- Develop an initial simplex tableau with artificial and slack variables in the basis and the other variables set equal to 0. compute the Z_j and $C_j - Z_j$ values for this tableau.
- Follow the five steps until an optimal solution has been reached

Review of Procedures for Solving LP Minimization Problems

1. Choose the variable with the negative $C_j - Z_j$ indicating the greatest improvement to enter the solution in the pivot column
2. Determine the row to be replaced and the pivot row by selecting the row with the smallest (nonnegative) quantity-to-pivot column substitution rate ratio
3. Calculate the new values for the pivot row
4. Calculate the new values for the other row(s)
5. Calculate the Z_j and $C_j - Z_j$ values for the tableau. If there are any $C_j - Z_j$ numbers less than 0, return to step 1. If not, and optimal solution has been reached.

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Infeasibility

- **Infeasibility** comes about when there is no solution that satisfies all of the problem's constraints
- In the simplex method, an infeasible solution is indicated by looking at the final tableau
- All $C_j - Z_j$ row entries will be of the proper sign to imply optimality, but an artificial variable will still be in the solution mix
- A situation with no feasible solution may exist if the problem was formulated improperly

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Special Cases

- We have seen how special cases arise when solving LP problems graphically
- They also apply to the simplex method
- You remember the four cases are
 - Infeasibility
 - Unbounded Solutions
 - Degeneracy
 - Multiple Optimal Solutions

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Infeasibility

- Illustration of infeasibility

C_j		\$5	\$8	\$0	\$0	\$M	\$M	
	SOLUTION MIX	X_1	X_2	S_1	S_2	A_1	A_2	QUANTITY
\$5	X_1	1	0	-2	3	-1	0	200
\$8	X_2	0	1	1	2	-2	0	100
\$M	A_2	0	0	0	-1	-1	1	20
	Z_j	\$5	\$8	-\$2	\$31 - M	-\$21 - M	\$M	\$1,800 + 20M
	$C_j - Z_j$	\$0	\$0	\$2	$M - 31$	$2M + 21$	\$0	

Table 9.11

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Unbounded Solutions

- **Unboundedness** describes linear programs that do not have finite solutions
- It occurs in maximization problems when a solution variable can be made infinitely large without violating a constraint
- In the simplex method this will be discovered prior to reaching the final tableau
- It will be manifested when trying to decide which variable to remove from the solution mix
- If all the ratios turn out to be negative or undefined, it indicates that the problem is unbounded

Unbounded Solutions

- The ratios from the pivot column

$$\begin{array}{l} \text{Ratio for the } X_2 \text{ row : } \frac{30}{-1} \\ \text{Ratio for the } S_2 \text{ row : } \frac{10}{-2} \end{array} \quad \begin{array}{l} \swarrow \\ \searrow \end{array} \quad \begin{array}{l} \text{Negative ratios} \\ \text{unacceptable} \end{array}$$

- Since both pivot column numbers are negative, an unbounded solution is indicated

Unbounded Solutions

- Problem with an unbounded solution

C_j →		\$6	\$9	\$0	\$0	
	SOLUTION MIX	X_1	X_2	S_1	S_2	QUANTITY
\$9	X_2	-1	1	2	0	30
\$0	S_2	-2	0	-1	1	10
	Z_j	-\$9	\$9	\$18	\$0	\$270
	$C_j - Z_j$	\$15	\$0	-\$18	\$0	

↑ Pivot column

Table 9.12

Degeneracy

- **Degeneracy** develops when three constraints pass through a single point
- For example, suppose a problem has only these three constraints $X_1 \leq 10$, $X_2 \leq 10$, and $X_1 + X_2 < 20$
- All three constraint lines will pass through the point (10, 10)
- Degeneracy is first recognized when the ratio calculations are made
- If there is a **tie** for the smallest ratio, this is a signal that degeneracy exists
- As a result of this, when the next tableau is developed, one of the variables in the solution mix will have a value of zero

Degeneracy

- Degeneracy could lead to a situation known as **cycling** in which the simplex algorithm alternates back and forth between the same nonoptimal solutions
- One simple way of dealing with the issue is to select either row in question arbitrarily
- If unlucky and cycling does occur, simply go back and select the other row

Degeneracy

- The ratios are computed as follows

For the X_2 row : $\frac{10}{0.25} = 40$

For the S_2 row : $\frac{20}{4} = 5$

For the S_3 row : $\frac{10}{2} = 5$

Tie for the smallest ratio indicates degeneracy

Degeneracy

- Problem illustrating degeneracy

C_j		\$5	\$8	\$2	\$0	\$0	\$0	
	SOLUTION MIX	X_1	X_2	X_3	S_1	S_2	S_3	QUANTITY
\$8	X_2	0.25	1	1	-2	0	0	10
\$0	S_2	4	0	0.33	-1	1	0	20
\$0	S_3	2	0	2	0.4	0	1	10
	Z_j	\$2	\$8	\$8	\$16	\$0	\$0	\$80
	$C_j - Z_j$	\$3	\$0	-\$6	-\$16	\$0	\$0	

↑ Pivot column

Table 9.13

Multiple Optimal Solutions

- In the simplex method, multiple, or alternate, optimal solutions can be spotted by looking at the final tableau
- If the $C_j - Z_j$ value is equal to 0 for a variable that is not in the solution mix, more than one optimal solution exists

Multiple Optimal Solutions

- A problem with alternate optimal solutions

C_j		\$3	\$2	\$0	\$0	
	SOLUTION MIX	X_1	X_2	S_1	S_2	QUANTITY
\$2	X_2	1.5	1	1	0	6
\$0	S_2	1	0	0.5	1	3
	Z_j	\$3	\$2	\$2	\$0	\$12
	$C_j - Z_j$	\$0	\$0	-\$2	\$0	

Table 9.14

High Note Sound Company Revisited

- You will recall the model formulation is

$$\begin{aligned} \text{Maximize profit} &= \$50X_1 + \$120X_2 \\ \text{subject to} \quad &2X_1 + 4X_2 \leq 80 \quad (\text{hours of electrician time}) \\ &3X_1 + 1X_2 \leq 60 \quad (\text{hours of technician time}) \end{aligned}$$

- And the optimal solution is

$$\begin{aligned} X_2 &= 20 \text{ receivers} \\ S_2 &= 40 \text{ hours slack in technician time} \\ X_1 &= 0 \text{ CD players} \\ S_1 &= 0 \text{ hours slack in electrician time} \end{aligned} \quad \left. \begin{array}{l} \text{Basic} \\ \text{variables} \end{array} \right\} \quad \left. \begin{array}{l} \text{Nonbasic} \\ \text{variables} \end{array} \right\}$$

Sensitivity Analysis with the Simplex Tableau

- Sensitivity analysis shows how the optimal solution and the value of its objective function change given changes in various inputs to the problem
- Computer programs handling LP problems of all sizes provide sensitivity analysis as an important output feature
- Those programs use the information provided in the final simplex tableau to compute ranges for the objective function coefficients and ranges for the RHS values
- They also provide “shadow prices,” a concept we will introduce in this section

High Note Sound Company Revisited

- High Note Sound Company graphical solution

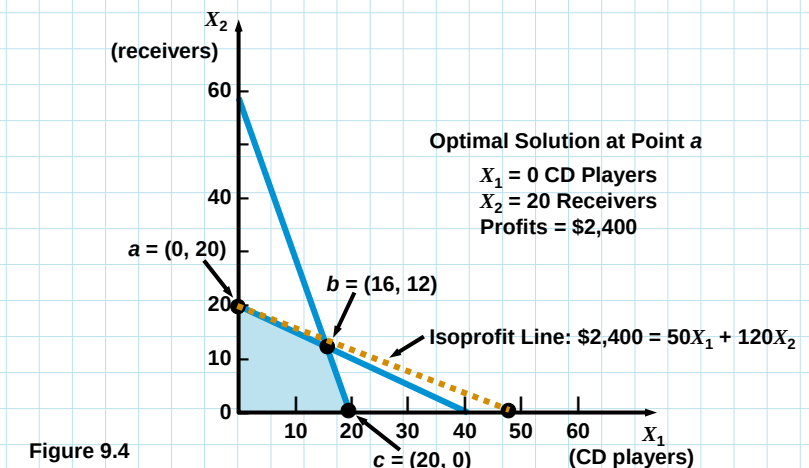


Figure 9.4

Changes in the Objective Function Coefficient

■ Optimal solution by the simplex method

C_j →		\$50	\$120	\$0	\$0	
	SOLUTION MIX	X_1	X_2	S_1	S_2	QUANTITY
\$120	X_2	0.5	1	0.25	0	20
\$0	S_2	2.5	0	-0.25	1	40
	Z_j	\$60	\$120	\$30	\$0	\$2,400
	$C_j - Z_j$	-\$10	\$0	-\$30	\$0	

Table 9.15

Changes in the Objective Function Coefficient

- This is a maximization problem so the basis will not change unless the $C_j - Z_j$ value of one of the nonbasic variables becomes greater than 0
- The values in the basis will not change as long as $C_j \leq Z_j$
- The solution will not change as long as X_1 does not exceed \$60 and the contribution rate of S_2 does not exceed \$30
- These values can also be made smaller without limit in this situation
- So the range of insignificance for the nonbasic variables is

$$-\infty \leq C_j \text{ (for } X_1) \leq \$60 \quad -\infty \leq C_j \text{ (for } S_1) \leq \$30$$

Changes in the Objective Function Coefficient

■ Nonbasic objective function coefficient

- The goal is to find out how sensitive the problem's optimal solution is to changes in the contribution rates of variables not currently in the basis
- How much would the objective function coefficients have to change before X_1 or S_1 would enter the solution mix and replace one of the basic variables?
- The answer lies in the $C_j - Z_j$ row of the final simplex tableau

Changes in the Objective Function Coefficient

■ Basic objective function coefficient

- Sensitivity analysis on objective function coefficients of variables in the basis or solution mix is slightly more complex
- A change in the profit or cost of a basic variable can affect the $C_j - Z_j$ values for *all* nonbasic variables
- That's because the C_j value is in both the row and column
- This then impacts the $C_j - Z_j$ row

Changes in the Objective Function Coefficient

- Consider a change in the profit contribution of stereo receivers
- The current coefficient is \$120
- The changed coefficient will be represented as Δ
- The revised final tableau will then be

C_j	SOLUTION MIX	X_1	X_2	S_1	S_2	QUANTITY
\$120 + Δ	X_2	0.5	1	0.25	0	20
\$0	S_2	2.5	0	-0.25	1	40
	Z_j	$\$60 + 0.5\Delta$	$\$120 + \Delta$	$\$30 + 0.25\Delta$	\$0	$\$2,400 + 20\Delta$
	$C_j - Z_j$	$-\$10 - 0.5\Delta$	\$0	$-\$30 - 0.25\Delta$	\$0	

Table 9.16

Changes in the Objective Function Coefficient

- Variable X_1 will not enter the basis unless the profit per receiver drops to \$100 or less
- For the S_1 column

$$-30 - 0.25\Delta \leq 0$$

$$-30 \leq 0.25\Delta$$

$$-120 \leq \Delta \text{ or } \Delta \geq -120$$

- Since the first inequality is more binding, we can say that the **range of optimality** for X_2 's profit coefficient is

$$\$100 \leq C_j (\text{for } X_2) \leq \infty$$

Changes in the Objective Function Coefficient

- The new $C_j - Z_j$ values in the table were determined in the same way as previous examples
- How may the value of Δ vary so that all $C_j - Z_j$ entries remain negative?
- To find out, solve for Δ in each column

$$-10 - 0.5\Delta \leq 0$$

$$-10 \leq 0.5\Delta$$

$$-20 \leq \Delta \text{ or } \Delta \geq -20$$

- This inequality means the optimal solution will not change unless X_2 's profit coefficient decreases by at least \$20, $\Delta = -20$

Changes in the Objective Function Coefficient

- In larger problems, we would use this procedure to test for the range of optimality of every real decision variable in the final solution mix
- Using this procedure helps us avoid the time-consuming process of reformulating and resolving the entire LP problem each time a small change occurs
- Within the bounds, changes in profit coefficients will not force a change in the optimal solution
- The value of the objective function will change, but this is a comparatively simple calculation

Changes in Resources or RHS Values

- Making changes in the RHS values of constraints result in changes in the feasible region and often the optimal solution
- **Shadow prices**
 - How much should a firm be willing to pay for one additional unit of a resource?
 - This is called the **shadow price**
 - Shadow pricing provides an important piece of economic information
 - This information is available in the final tableau

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Changes in Resources or RHS Values

- An important property of the $C_j - Z_j$ row is that the negatives of the numbers in its slack variable (S_i) columns provide us with shadow prices
- A **shadow price** is the change in value of the objective function from an increase of one unit of a scarce resource
- High Note Sound is considering hiring an extra electrician at \$22 per hour
- In the final tableau we see S_1 (electricians' time) is fully utilized and has a $C_j - Z_j$ value of $-\$30$
- They should hire the electrician as the firm will **net** \$8 ($= \$30 - \22)

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Changes in Resources or RHS Values

- Final tableau for High Note Sound

C_j		\$50	\$120	\$0	\$0	
	SOLUTION MIX	X_1	X_2	S_1	S_2	QUANTITY
\$120	X_2	0.5	1	0.25	0	20
\$0	S_2	2.5	0	-0.25	1	40
	Z_j	\$60	\$120	\$30	\$0	\$2,400
	$C_j - Z_j$	-\$10	\$0	-\$30	\$0	

Objective function increases by \$30 if 1 additional hour of electricians' time is made available

Table 9.17

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Changes in Resources or RHS Values

- Should High Note Sound hire a part-time audio technician at \$14 per hour?
- In the final tableau we see S_2 (audio technician time) has slack capacity (40 hours) a $C_j - Z_j$ value of \$0
- Thus there would be no benefit to hiring an additional audio technician

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Changes in Resources or RHS Values

- **Right-hand side ranging**
 - We can't add an unlimited amount of a resource without eventually violating one of the other constraints
 - **Right-hand-side ranging** tells us how much we can change the RHS of a scarce resource without changing the shadow price
 - Ranging is simple in that it resembles the simplex process

Changes in Resources or RHS Values

- The smallest negative ratio (–160) tells us the number of hours that can be added to the resource before the solution mix changes
- In this case, that's 160 hours
- So the range over which the shadow price for electricians' time is valid is 0 to 240 hours
- The audio technician resource is slightly different
- There is slack in this resource ($S_2 = 40$) so we can reduce the amount available by 40 before a shortage occurs
- However, we can increase it indefinitely with no change in the solution

Changes in Resources or RHS Values

- This table repeats some of the information from the final tableau for High Note Sound and includes the ratios

QUANTITY	S_1	RATIO
20	0.25	$20/0.25 = 80$
40	–0.25	$40/-0.25 = -160$

- The smallest positive ratio (80 in this example) tells us how many hours the electricians' time can be reduced without altering the current solution mix

Changes in Resources or RHS Values

- The substitution rates in the slack variable column can also be used to determine the actual values of the solution mix variables if the right-hand-side of a constraint is changed using the following relationship

$$\text{New quantity} = \text{Original quantity} + \left(\text{Substitution rate} \right) \left(\text{Change in the RHS} \right)$$

Changes in Resources or RHS Values

- For example, if 12 more electrician hours were made available, the new values in the quantity column of the simplex tableau are found as follows

ORIGINAL QUANTITY	S_1	NEW QUANTITY
20	0.25	$20 + 0.25(12) = 23$
40	-0.25	$40 + (-0.25)(12) = 37$

- If 12 hours were added, $X_2 = 23$ and $S_2 = 37$
- Total profit would be $50(0) + 120(23) = \$2,760$, an increase of \$360
- This of course, is also equal to the shadow price of \$30 times the 12 additional hours

Sensitivity Analysis by Computer

Here is the solution: Profit = \$2,400.

By selecting Sensitivity, we can produce Program 9.1B.

This formulation is shown in detail as Program 7.6A.

Solver Results
Solver found a solution. All constraints and optimality conditions are satisfied.

Reports
Answer
Sensitivity
Limits

☒ Keep Solver Solution
☐ Restore Original Values

OK Cancel Save Scenario... Help

Program 9.1a

Sensitivity Analysis by Computer

- Solver in Excel has the capability of producing sensitivity analysis that includes the shadow prices of resources
- The following slides present the solution to the High Note Sound problem and the sensitivity report showing shadow prices and ranges

Sensitivity Analysis by Computer

Microsoft Excel 8.0 Sensitivity Report
Worksheet: [captures.xls]7.8
Report Created: 8/13/98 10:54:59 AM

Adjustable Cells

We will use 80 hours of electrician time and 20 hours of audio technician time.

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$4	Value CD Players	0	-10	50	10	1E+30
\$C\$4	Value Receivers	20	0	120	1E+30	20

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$D\$9	Electrician hours Used	80	30	80	160	80
\$D\$10	Audio technician hours Used	20	0	60	1E+30	40

Program 9.1b

The Dual

- Every LP problem has another LP problem associated with it called the **dual**
- The first way of stating a problem (what we have done so far) is called the **primal**
- The second way of stating it is called the **dual**
- The solutions to the primal and dual are equivalent, but they are derived through alternative procedures
- The dual contains economic information useful to managers and may be easier to formulate

The Dual

- Illustrating the **primal-dual relationship** with the High Note Sound Company data
- The primal problem is to determine the best production mix between CD players (X_1) and receivers (X_2) to maximize profit

$$\begin{aligned} \text{Maximize profit} &= \$50X_1 + \$120X_2 \\ \text{subject to} \quad &2X_1 + 4X_2 \leq 80 \quad (\text{hours of available electrician time}) \\ &3X_1 + 1X_2 \leq 60 \quad (\text{hours of audio technician time available}) \end{aligned}$$

The Dual

- Generally, if the LP primal is a maximize profit problem with less-than-or-equal-to resource constraints, the dual will involve minimizing total opportunity cost subject to greater-than-or-equal-to product profit constraints
- Formulating a dual problem is not complex and once formulated, it is solved using the same procedure as a regular LP problem

The Dual

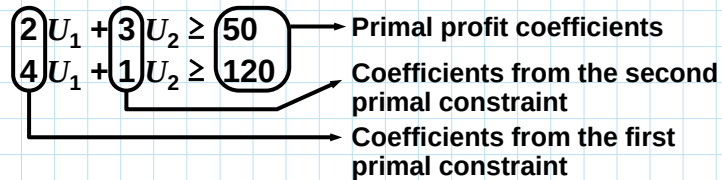
- The dual of this problem has the objective of minimizing the opportunity cost of not using the resources in an optimal manner
- The variables in the dual are
 - U_1 = potential hourly contribution of electrician time, or the dual value of 1 hour of electrician time
 - U_2 = the imputed worth of audio technician time, or the dual of technician resource
- Each constraint in the primal problem will have a corresponding variable in the dual and each decision variable in the primal will have a corresponding constraint in the dual

The Dual

- The RHS quantities of the **primal** constraints become the dual's **objective function** coefficients
- The total opportunity cost will be represented by the function

$$\text{Minimize opportunity cost} = 80U_1 + 60U_2$$

- The corresponding dual constraints are formed from the transpose of the primal constraint coefficients



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Steps to Form the Dual

- If the primal is a maximization problem in the standard form, the dual is a minimization, and vice versa
- The RHS values of the primal constraints become the dual's objective coefficients
- The primal objective function coefficients become the RHS values of the dual constraints
- The transpose of the primal constraint coefficients become the dual constraint coefficients
- Constraint inequality signs are reversed

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The Dual

- The first constraint says that the total imputed value or potential worth of the scarce resources needed to produce a CD player must be at least equal to the profit derived from the product
- The second constraint makes an analogous statement for the stereo receiver product

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Solving the Dual of the High Note Sound Company Problem

- The formulation can be restated as

$$\begin{aligned} &\text{Minimize} \\ &\text{opportunity cost} = 80U_1 + 60U_2 + 0S_1 + 0S_2 + MA_1 + MA_2 \\ &\text{subject to:} \quad \begin{array}{ccccccc} 2U_1 & + & 3U_2 & - & 0S_1 & & + & 1A_1 & & = & 50 \\ 4U_1 & + & 1U_2 & & - & 0S_2 & & & + & 1A_2 & = & 120 \end{array} \end{aligned}$$

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Solving the Dual of the High Note Sound Company Problem

■ The first and second tableaus

		C_j		80	60	0	0	M	M	
			SOLUTION MIX	U_1	U_2	S_1	S_2	A_1	A_2	QUANTITY
First tableau	\$M	A_1		2	3	-1	0	1	0	50
	\$M	A_2		4	1	0	-1	0	1	120
		Z_j		\$6M	\$4M	-\$M	-\$M	\$M	\$M	\$170M
		$C_j - Z_j$		$80 - 6M$	$60 - 4M$	M	M	0	0	
Second tableau	\$80	U_1		1	1.5	-0.5	0	0.5	0	25
	\$M	A_2		0	-5	2	-1	-2	1	20
		Z_j		\$80	$\$120 - 5M$	$-\$40 + 2M$	-\$M	$\$40 - 2M$	\$M	$\$2,000 + 20M$
		$C_j - Z_j$		0	$5M - 60$	$-2M + 40$	M	$3M - 40$	0	

Table 9.18

Solving the Dual of the High Note Sound Company Problem

- In the final simplex tableau of a primal problem, the absolute values of the numbers in the $C_j - Z_j$ row under the slack variables represent the solutions to the dual problem
- They are shadow prices in the primal solution and marginal profits in the dual
- The absolute value of the numbers of the $C_j - Z_j$ values of the slack variables represent the optimal values of the primal X_1 and X_2 variables
- The maximum opportunity cost derived in the dual must always equal the maximum profit derived in the primal

Solving the Dual of the High Note Sound Company Problem

■ Comparison of the primal and dual optimal tableaus

		C_j		\$50	\$120	\$0	\$0	
			Solution Mix	X_1	X_2	S_1	S_2	Quantity
\$120	X_2			0.5	1	0.25	0	20
\$0	S_2			2.5	0	-0.25	1	40
	Z_j			60	120	30	0	\$2,400
	$C_j - Z_j$			-10	0	-30	0	

		C_j		80	60	0	0	M	M	
			Solution Mix	U_1	U_2	S_1	S_2	A_1	A_2	Quantity
80	U_1			1	0.25	0	-0.25	0	0.5	30
0	S_1			0	-2.5	1	-0.5	-1	0.25	10
	Z_j			80	20	0	-20	0	20	\$2,400
	$C_j - Z_j$			0	40	0	20	M	$M - 20$	

Figure 9.5

Karmakar's Algorithm

- In 1984, Narendra Karmakar developed a new method of solving linear programming problems called the **Karmakar algorithm**
- The simplex method follows a path of points on the outside edge of feasible space
- Karmakar's algorithm works by following a path a points **inside** the feasible space
- It is much more efficient than the simplex method requiring less computer time to solve problems
- It can also handle **extremely** large problems allowing organizations to solve previously unsolvable problems